Fairness and Efficiency in Random Assignment

Three Impossibility Results

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Fairness and Efficiency...

- ...in Economy:
  - larger pie vs. fair division (optimal distribution, growth etc.)
- ...in Matching and Random Assignment:
  - satisfaction with the final assignment
  - fairness: stability, litigation potential

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Fairness and Efficiency in Matching and Random Assignment

- ...Ex-ante: Hylland and Zeckhauser (1979), Zhou (1990), Bogomolnaia and Moulin (2001)
- ...Ex-post: Abdulkadiroglu and Sönmez (2003), Kesten and Yazici (2010)
- ...in Large markets: Che and Kojima (2010), Kojima and Manea (2010), Budish (2011), Che and Tercieux (2014)

House allocation problem

- Optimal resource allocation:
  - indivisible goods, no transfers, no endowments
- $A = \{a_1, a_2, ..., a_N\}$ - agents, $H = \{h_1, h_2, ..., h_N\}$ - houses
- Preference profile - set of strict preferences over H: $\succ = (\succ_a)_{a \in A}$
- Matching μ - deterministic solution
- Random assignment P - stochastic solution
  - always representable as a lottery over matchings (Birkhoff-von Neumann Theorem)
- Mechanism - systematic procedure φ: $\succ \longrightarrow P$
- Example: Random serial dictatorship (RSD), (aka Random priority)

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Strategy-proofness and Efficiency

- Strategy-proofness (SP) - nobody can benefit from misreporting
  - $\varphi(\succ) \geq_a \varphi_a(\succ'_a, \succ_{-a})$, ($\geq_a$ – F.O.S.D.)
  - focus on strategy-proofness; RSD
- Efficiency - no other (random) assignment dominates
  - Matching μ is Pareto efficient
  - P is ex-post efficient (ExPE) if representable as a lottery over efficient matchings
  - P is ordinally efficient (OE) if no other P' dominates at $\succ$ (F.O.S.D.)
  - RSD is ex-post efficient but not ordinally efficient

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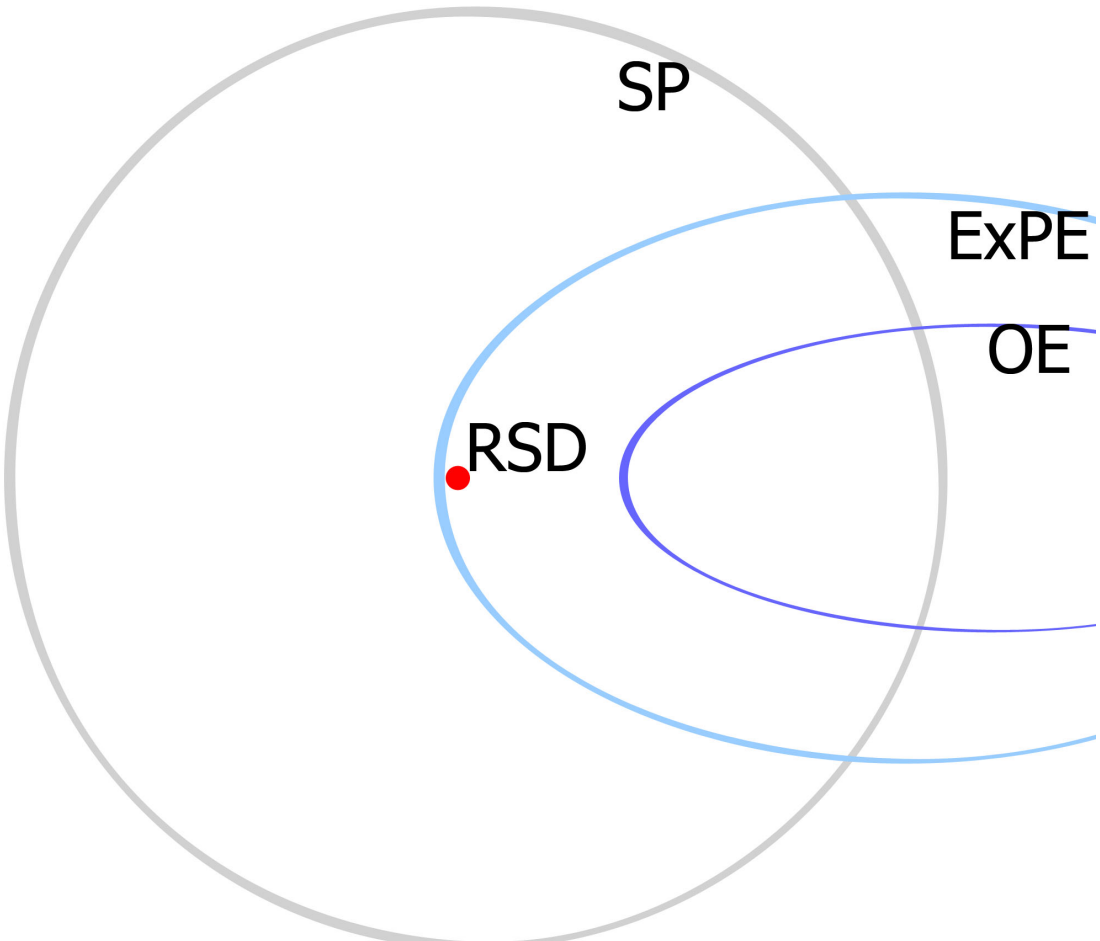
Efficiency (N>3)



Fairness

- Fairness - no agent is worse off as compared to (some) others or to some alternative assignment
- Envy-freeness (EF) - each agent prefers her own random assignment
  - possible, "subjective" envy
- Weak envy-freeness (wEF) - no agent **strictly** prefers someone's random assignment
  - inevitable, "objective" (and verifiable) envy
- Equal treatment of equals (ETE) - agents with identical preferences get identical random assignments
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Fairness: Example

	h_1	h_2	h_3
a_1	.5	.0	.5
a_2	.3	.4	.3
a_3	.2	.6	.2

, all agents prefer h_1 to h_2, and h_2 to h_3

- −Envy-freeness (EF) - agent might prefer other's random assignment
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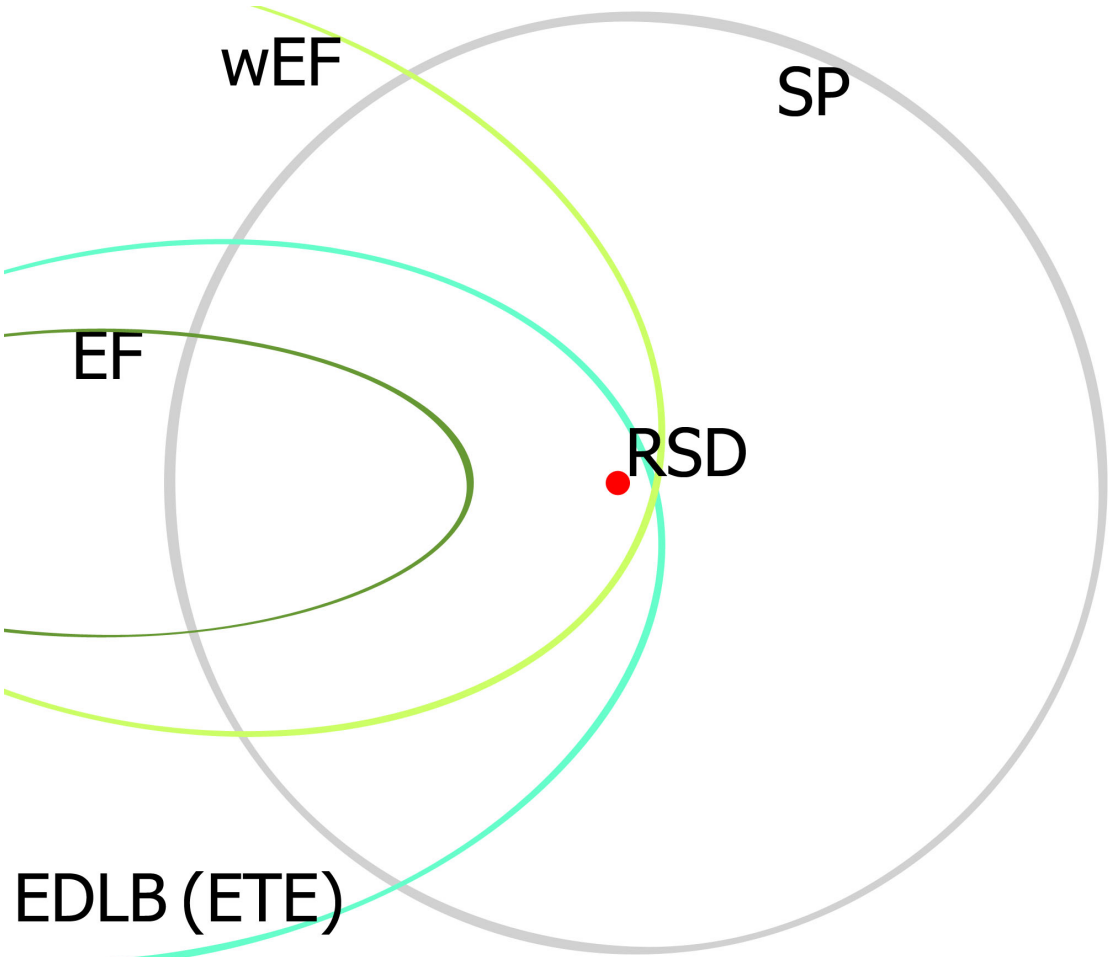
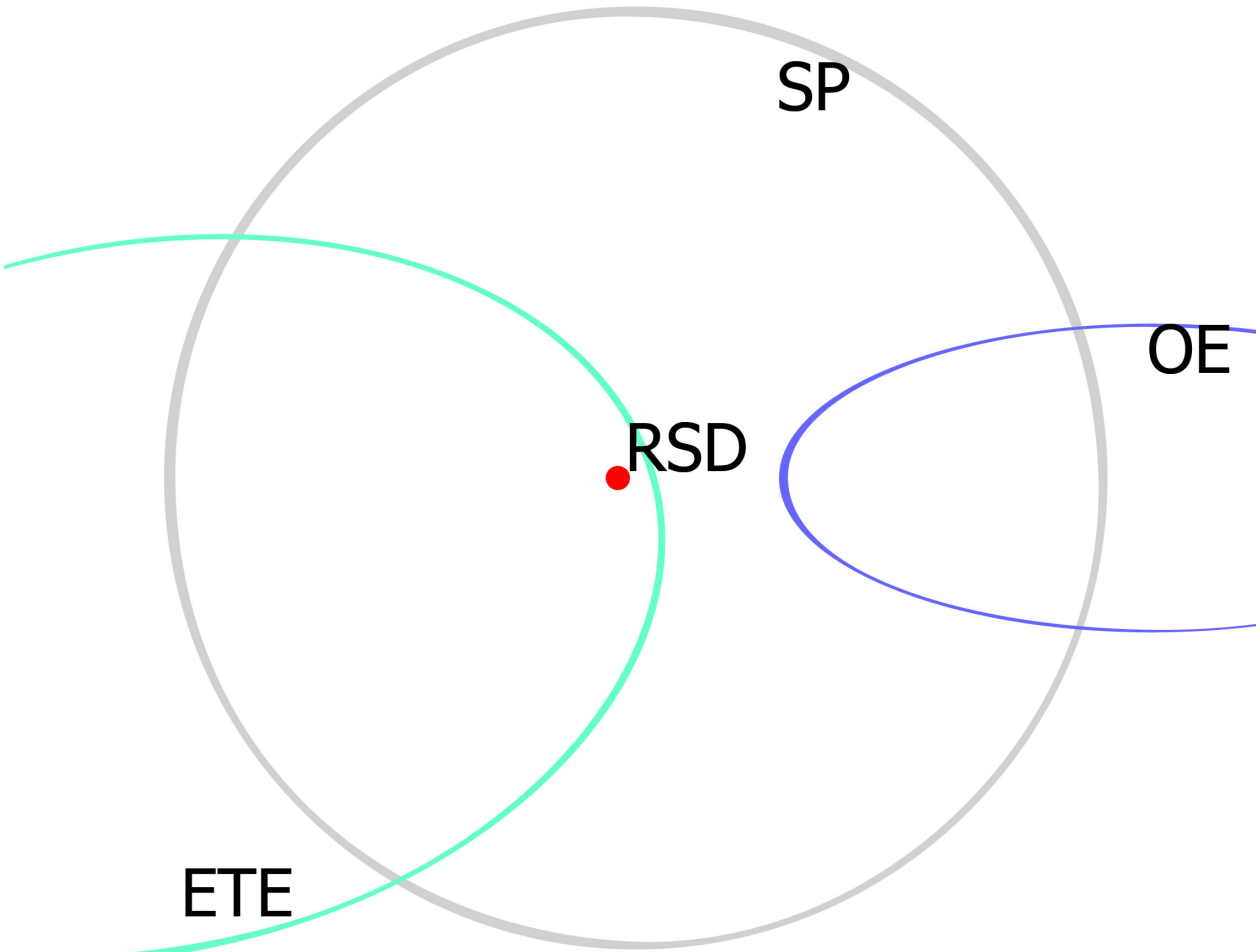
Fairness (N>2)

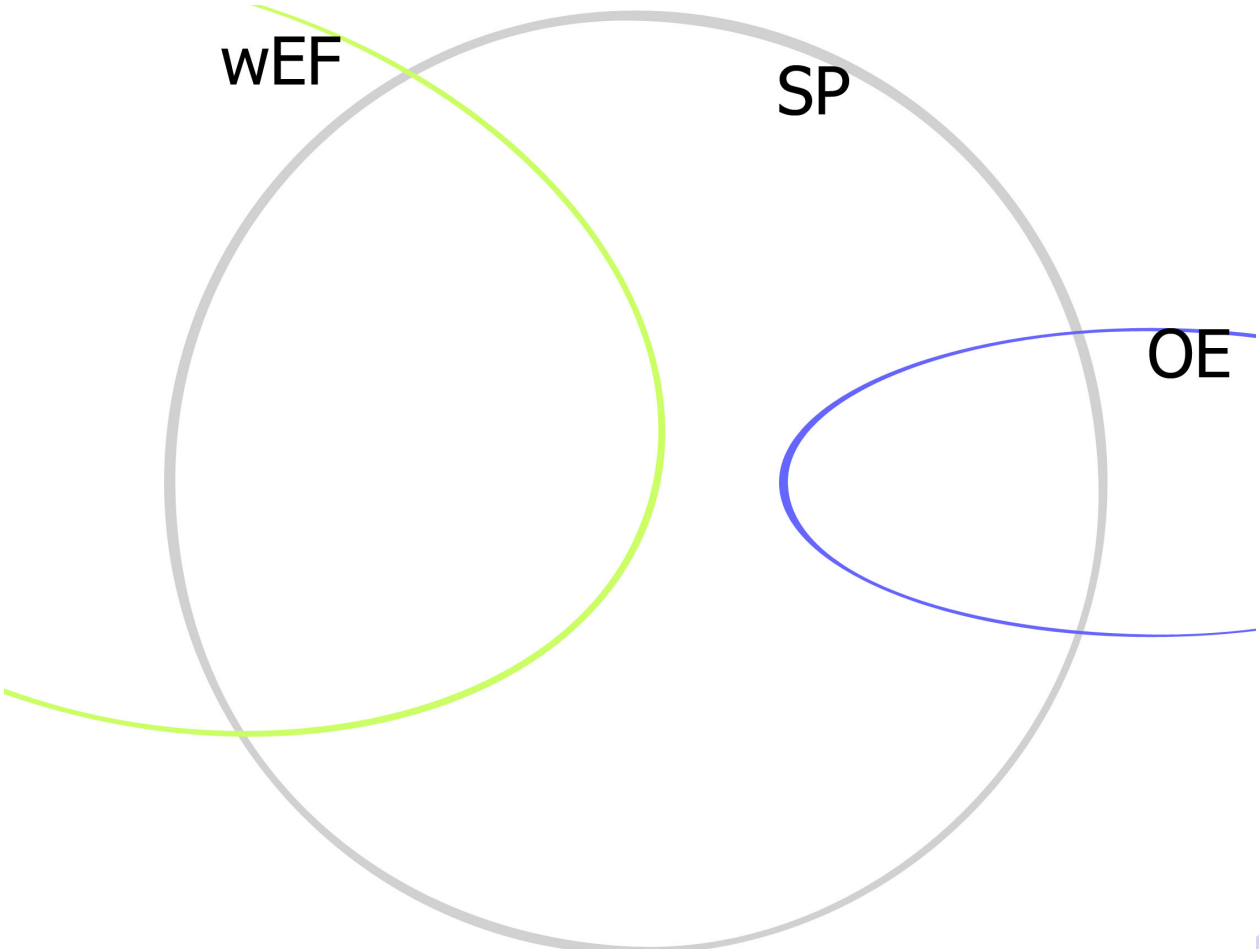


Previous result: Bogomolnaia and Moulin'2001

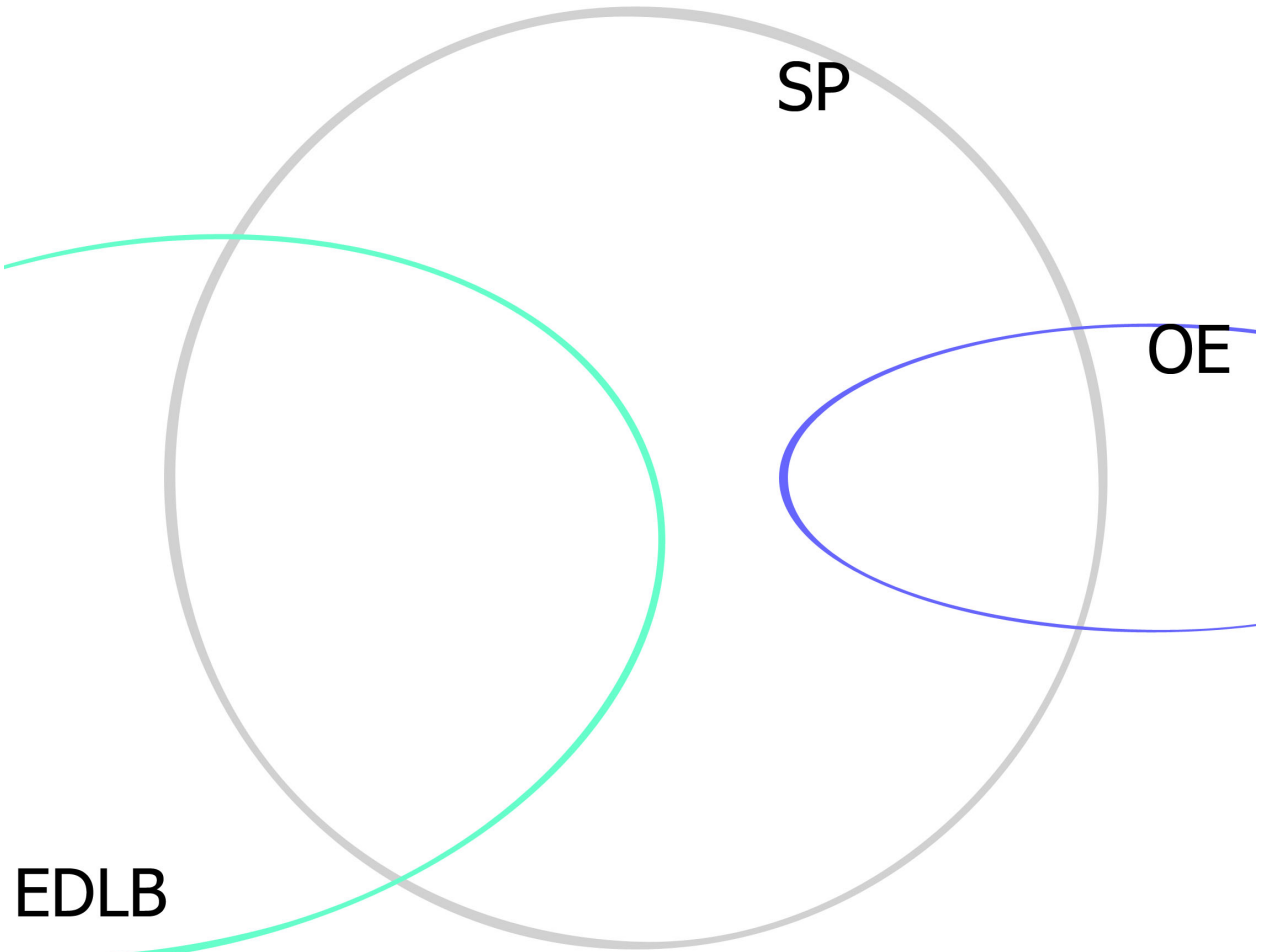
No ordinally-efficient, strategy-proof, ETE mechanism



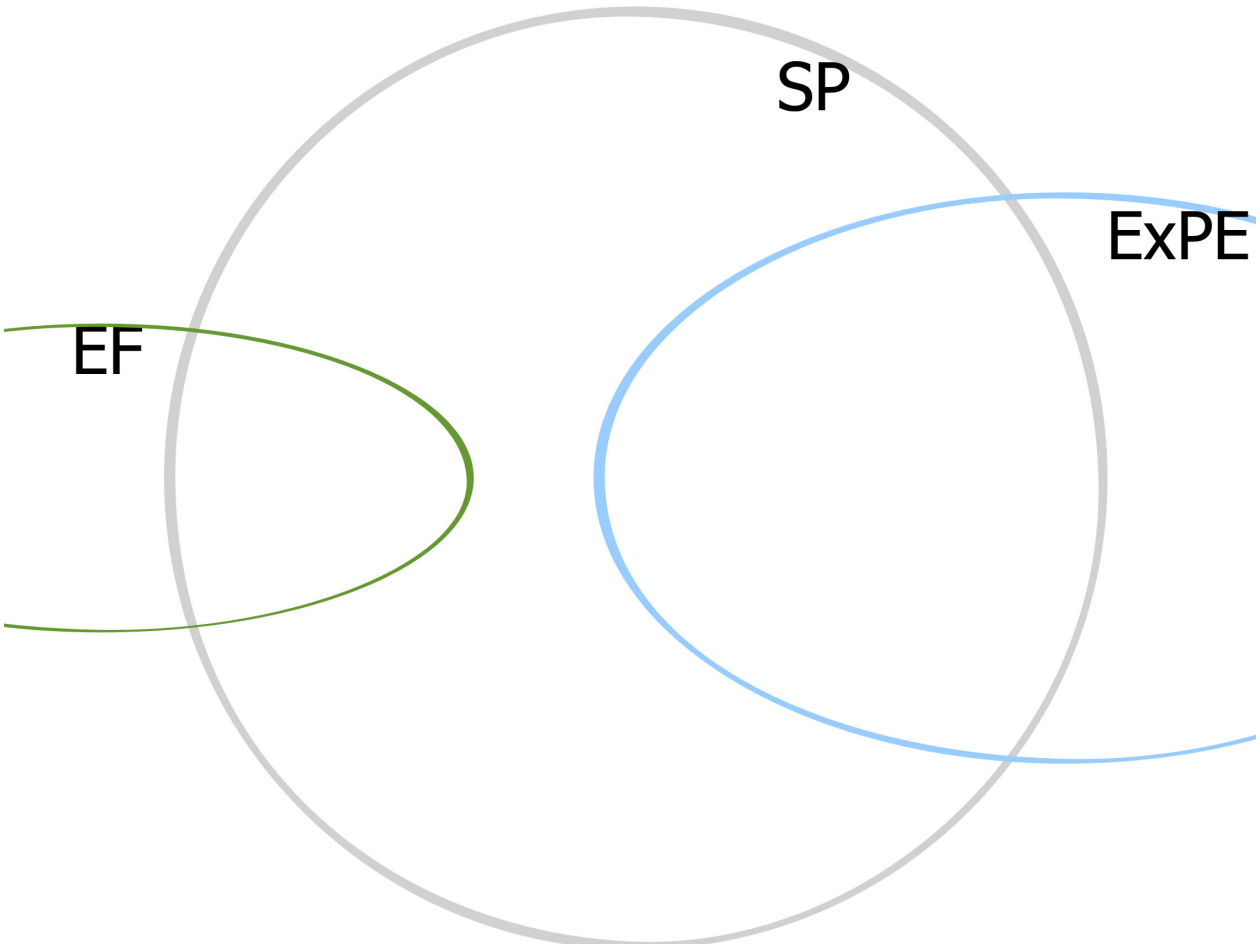
No weakly ordinally efficient, strategy-proof, weakly envy-free mechanism



No ordinally efficient, strategy-proof, EDLB mechanism



No ex-post efficient, strategy-proof, envy-free mechanism



Result 3: sketch of the proof

- N=3: Ex-post efficiency, envy-freeness and strategy-proofness are mutually incompatible
- $\succ$: $\begin{array}{c|ccc} a_1 & h_1 & h_2 & h_3 \\ a_2 & h_1 & h_3 & h_2 \\ a_3 & h_2 & h_1 & h_3 \end{array}$, $\varphi(\succ) = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix}$.
- $\succ'$: $\begin{array}{c|ccc} a_1 & h_1 & h_2 & h_3 \\ a_2 & h_1 & h_3 & h_2 \\ a_3 & h_1 & h_2 & h_3 \end{array}$, $\varphi(\succ') = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}$
- N>3: consider some preference profile with block preferences
  - agents a_1, a_2, a_3 as before
  - agents $a_{j>3}$ prefer houses h_j

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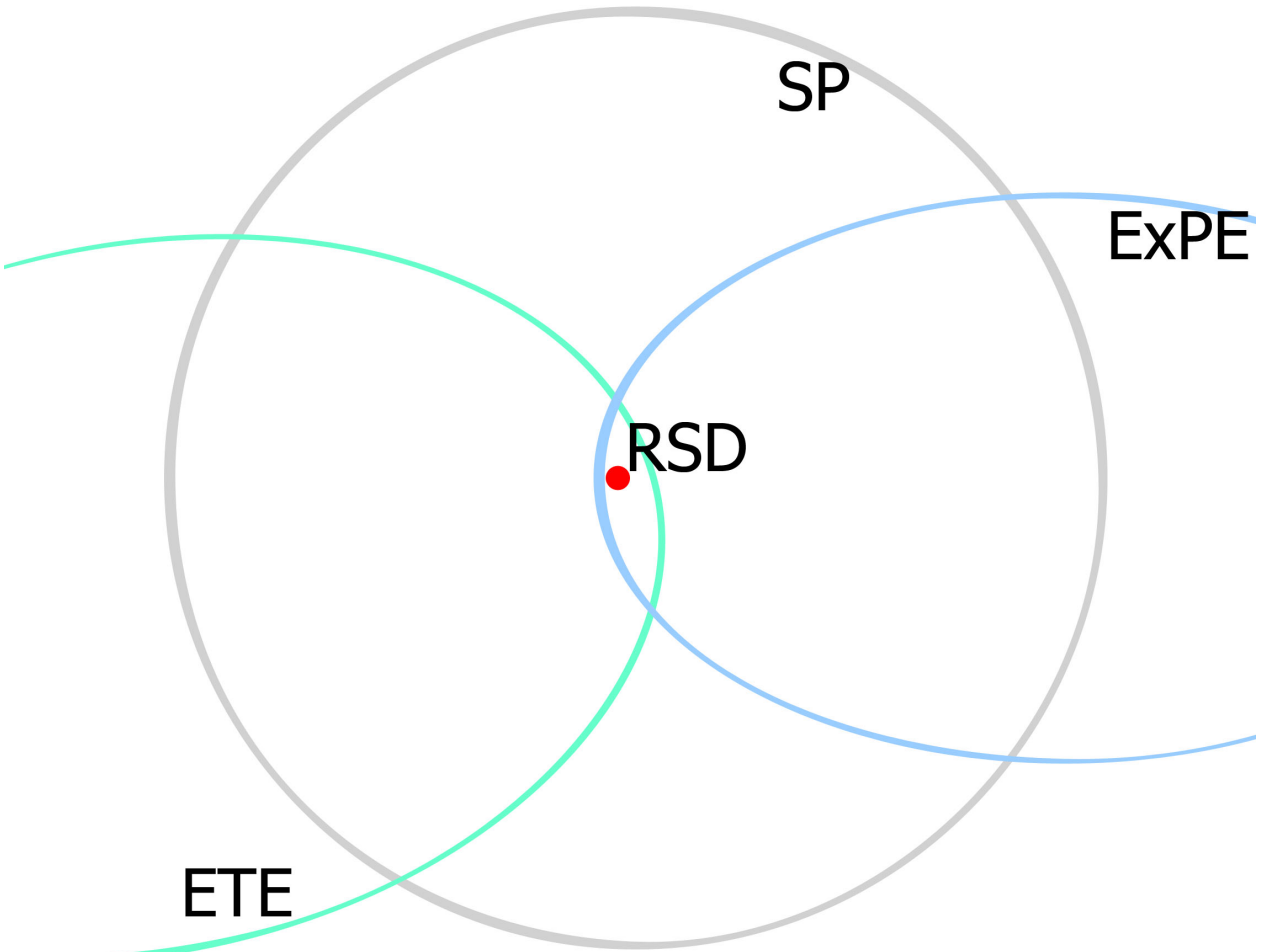
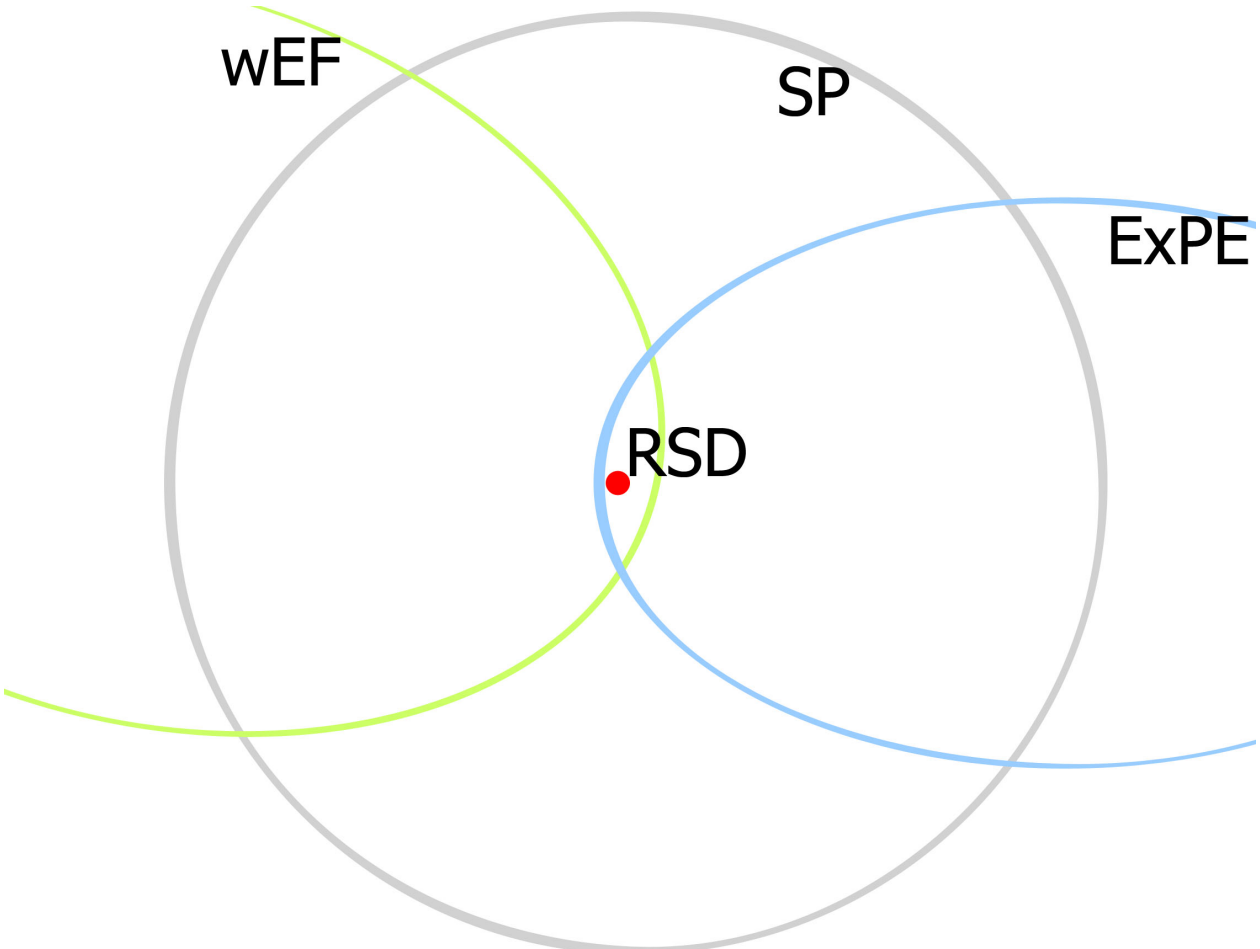
For N=3 ExPE+ETE+SP=RSD (BM'2001)



For N=3 ExPE+wEF+SP=RSD (Proposition 3)



Summary of the results

		Strategy-proof mechanisms			
		Envy-free	Weak-envy-free	Equal division lower bound	Equal treatment of equals
Ex-post efficient	$N = 3$	$\emptyset$ (Result 3, BM*)	RSD!	RSD	RSD! (BM)
	$N > 3$	$\emptyset$ (Result 3)	RSD (BM)	RSD	RSD
Ordinally efficient	$N > 3$	$\emptyset$	$\emptyset$ (Result 1)	$\emptyset$ (Result 2)	$\emptyset$ (BM)

- Implications:
  - Results 1,2: similar to Bogomolnaja and Moulin but with more relevant fairness notions,
  - Result 2: no fair exchange mechanism can be ordinally efficient and strategy-proof (including pseudo-market mechanisms),
  - Result 3: relevant for applications - uses ex-post efficiency and strategy-proofness (SD- and TTC-based mechanisms),
  - Alternative characterization of RSD: SP, ExPE, wEF,
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