

Welfare-Maximizing Assignment of Agents to Hierarchical Positions

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March 20, 2014

Abstract

We consider an environment where agents must be allocated to one of three kinds of hierarchical positions with limited capacities. No monetary transfers are allowed. We assume that agents' payoffs for being assigned to medium positions are their private information, and we consider bayesian incentive-compatible direct mechanisms. We solve for utilitarian and Rawlsian welfare-maximizing rules. Interestingly, the two optimal mechanisms are implemented by Hylland and Zeckhauser (1979)'s pseudomarket mechanism with equal budgets (PM) and the Boston mechanism without priorities (BM), for a variety of cases. When the market is tough (i.e., when medium positions are overdemand), then utilitarian optimal, Rawlsian optimal, PM, and BM assignments coincide. Otherwise, when the market is mild, PM and BM differ, and each one implements the two optimal mechanisms under different assumptions on the curvature of the payoff distribution of the medium positions. When we allow medium positions to be the favorite for some agent types, PM and BM may still be optimal in tough markets, and a bias in favor of BM rather than PM appears in mild markets.

Keywords: Optimal mechanism, no transfers, assignment, pseudomarket, Boston mechanism.

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1 Introduction

In now a seminal paper, Hylland and Zeckhauser (1979) take a market design perspective on the problem of assigning agents to positions, where monetary transfers are not allowed and each agent is eventually assigned to exactly one position. Their model is motivated by important real-world applications such as assignment of legislators to committees, faculty members to offices, students to public schools, and MBA students to different courses. These environments can be modeled as “assignment games” in which (i) agents and positions are indivisible (yet probabilistic assignment to positions are allowed); (ii) preferences of agents over positions may be their private information, and hence agents can respond to mechanisms strategically rather than truthfully, and; (iii) a medium of exchange such as money, for various reasons, is not an acceptable instrument. Hylland and Zeckhauser (1979) introduce the “pseudomarket mechanism,” in which the individuals use “fake money” to buy assignment probabilities for different kinds of positions. The pseudomarket has good efficiency properties. More specifically, it is “ex-ante” efficient in the sense that there cannot be any other probabilistic assignment that would make all the agents better off.¹

In this paper, we take an optimal mechanism design approach to the same problem. To make the model tractable, we study a continuum economy (with unit mass of players and positions) where three kinds of positions are ordinally ranked the same way by all agents.² For instance, we can imagine a society that cannot provide the same social status to all of its members. Naturally, every member prefers belonging to the highest class and dislikes being relegated to the lowest social class. However, agents differ in several dimensions of personality, and this determines some differences in the intensity of their preferences. Risk-averse or low-ambition agents might prefer belonging to a medium class with certainty rather than facing a lottery over either being assigned to a high-class position or being assigned to a low-class position. The opposite would be true for less risk-averse or more ambitious agents. This postulates a constitutional design problem. How do we distribute agents to the available social positions so that we optimize a given notion of social welfare?

We focus on two notions of optimality: Utilitarian (maximizing the unweighted average of all agents’ expected payoffs) and Rawlsian (maximizing the minimum payoff). We characterize the optimal incentive-compatible random assignment rule under either notion of

¹Moreover, it does not generate envy when all agents face the same budget limit. The converse is true in atomless economies with a continuum of agents, from Thomson and Zhou (1993): any efficient and envy-free random assignment can be obtained through a Pseudomarket equilibrium with equal budgets.

²In the discussion section we also consider the case where agents ordinal preferences may be different from each other.

optimality. Interestingly, we find that the mechanisms implementing the optimal assignment rules are widely known in the literature on matching and assignment problems. Either the pseudomarket with equal budgets (PM) or the widely debated Boston mechanism³ with no preexisting priority rights (BM) turns out to be optimal for a variety of cases, depending on the distribution of cardinal preferences.

In our main result (Theorem 16), we establish this surprising result. We find that when the market is *tough* in the sense of potential excess demand for top- and middle-class positions, the utilitarian-optimal assignment rule, the Rawlsian-optimal assignment rule, the PM competitive equilibrium allocation, and the BM Bayesian Nash-equilibrium allocation are identical. This is true for all value distributions F . Miralles (2008) previously established that when there is only one underdemanded kind of position, BM and PM obtain the same allocation. What is new in our result is that this allocation is both utilitarian- and Rawlsian-optimal.⁴ Moreover, we show that when the market is *mild* as opposed to *tough*, it turns out that the nature of the optimal allocation rule depends on the curvature of F . More specifically, we give sufficient conditions on the value distribution F that makes either PM or BM optimal in the utilitarian and Rawlsian maximization problems.⁵

From a methodological point of view, our paper solves a more complex mathematical problem than a standard auction design problem, for at least two reasons. First, the “numeraire” good, in our case the probability of being assigned to top-class positions, is globally constrained, since the capacity for top positions is limited. Second, the probability of being assigned to top- and middle-class positions must satisfy proper probability constraints for each agent. Namely, no probability can be negative, and the sum of these two probabilities cannot exceed 1. The most closely related paper on this methodological issue is Miralles (2012). The key feature that is different in the present paper is that every agent must end up in exactly one position, whereas Miralles (2012) left the number of finally obtained objects unconstrained.

To cope with interpersonal comparability, we adopt a double normalization such that being assigned to top positions gives a utility of one, while being assigned to low-class positions gives zero utility. Being assigned a middle-class position gives a utility $v \in [0, 1]$,

³See Abdulkadiroğlu and Sönmez (2003), Ergin and Sönmez (2006), and Pathak and Sönmez (2008), among others.

⁴This is not a trivial equivalence. Example 21 in Subsection 4.2 shows that not every weighted average of agents’ payoffs is maximized by these mechanisms in a tough market.

⁵Example 23 in Subsection 4.4 shows that for mild markets, neither PM nor BM may be optimal in the Utilitarian or Rawlsian maximization problems for some specifications of the parameters.

which is the private information of the agents.⁶ In other words, we measure the agent's *degree of success*, where getting the most-preferred position entails total success, getting a low-class position entails total failure, and obtaining a middle position implies a variable degree of success, depending on how the agent values it. The distribution function of this parameter, $F(\cdot)$, with positive, differentiable density $f(\cdot)$, characterizes the demand side of this economy. We make the standard auction theory assumptions that both the “seller”- and “buyer”-virtual valuations (respectively, $H(v) = v + F(v)/f(v)$ and $J(v) = v - (1 - F(v))/f(v)$) are increasing functions. The supply side is characterized by the limited capacities for high- and middle-class positions, respectively μ_1 and μ_2 , where $\mu_1, \mu_2, \mu_1 + \mu_2 \in (0, 1)$.

We first observe the *relaxed* problem that ignores the following families of constraints: for each agent, (1) no assigned probability can be negative, and (2) the sum of all probabilities must be 1. We find that the optimal (in both the utilitarian and the Rawlsian sense) assignment rule of the relaxed problem does not violate these previously ignored constraints *if and only if* $\mu_1 - (1 - \mu_2)F^{-1}(1 - \mu_2) = 0$. Observe that the left-hand side (LHS) is increasing in both capacities, μ_1 and μ_2 . It also decreases if the distribution of valuations for middle-class positions becomes stronger in the first-order stochastic sense. In sum, tougher competitive conditions (both on the demand and on the supply sides) reduce the value of this LHS. This sets the specific *boundary between a tough market and a mild market* in our paper: in the *tough* market, the LHS is nonpositive, whereas in a *mild* market, the LHS is positive. We treat each case differently.

We use standard manipulations of incentive-compatibility constraints to reduce the impact of the aforementioned families of additional constraints. For $p(v)$ denoting the probability that a v -type agent is assigned to top positions and $q(v)$ denoting her probability of being assigned to middle-class positions, incentive compatibility implies that $p(\cdot) + q(\cdot)$ is increasing and $p(\cdot)$ is decreasing. So the previous families of constraints are reduced to two single constraints: the *upper-bound (UB)* constraint $p(1) + q(1) \leq 1$ and the *lower-bound (LB)* constraint $p(1) \geq 0$ (Proposition 3). In a tough market, the optimal rule implies that all agents must bear some risk of ending up in a low-class position. In this case, it turns out that LB binds at the optimal solution. On the other hand, in a mild market, high- v agents optimally obtain certain allocation to middle-class positions or better, and in the optimal solution UB binds.

In a tough market, we first show that UB cannot bind. This implies that an agent

⁶The main reason to consider only private information for medium positions is to be able to solve a one-dimensional private information mechanism design problem. The multidimensional problem is known to be thorny even for environments with transfers.

with a top preference for middle-class positions ($v = 1$) cannot be assigned there or better with certainty. Among the assignment rules abiding by this, optimality is obtained with a two-step rule characterized by a cutoff d such that all high-class positions are assigned to types below d and all middle-class positions are assigned to types above d (Proposition 6). Interestingly, it turns out that this rule coincides with the equilibrium allocation both in PM and BM when only low-class positions are underdemanded (Theorem 16).

On the other hand, in a mild market, we first study two-step rules such that types below some cutoff c receive a probability bundle while types above receive another bundle. Given the UB constraint and the feasibility constraint, a two-step mechanism is fully characterized by this cutoff c with valid values along a closed interval (Proposition 8). The lowest possible c happens to characterize the equilibrium allocation of PM, while the highest possible c characterizes BM. Whether the objective function is increasing or decreasing in c in the utilitarian case depends on the concavity/convexity of H (the seller's virtual valuation). We show that no other feasible incentive-compatible allocation rule can utilitarian-dominate PM (BM) when H is strictly concave (convex). For the Rawlsian problem, to the concavity/convexity of H we need to add conditions on whether the uniform distribution hazard-rate dominates/is hazard-rate dominated by F , in order to show that either PM or BM is optimal (Theorem 16).

The two optimal mechanisms have properties in common in that they both provide market-like trade-offs. PM is a competitive market that is designed to close in one round, since agents are constrained to buy a proper probability distribution. At other extreme, BM can be understood as a multi-round competitive market. In each round, agents who were unassigned at the end of the previous round obtain a new budget with which they buy a probability bundle that does not add up to more than 1. In both cases, spending money on one position can reduce the chances for other positions. This is in sharp contrast with Gale and Shapley's (1962) well-known deferred acceptance (DA) algorithm, where betting on the most-preferred option does not harm the agent's chances at other positions. In this context with perfectly aligned ordinal preferences, it is known that DA implements a Pareto-pessimal assignment rule (Miralles, 2008; Abdulkadiroğlu, Che and Yasuda, 2011).⁷

Since there are no preexisting priority rights, DA coincides in its allocation with random serial dictatorship, which is also equivalent to top-trading cycles from random allocation (Abdulkadiroğlu and Sönmez, 1998). Moreover, DA coincides with Bogolmonaia and Moulin's

⁷Every other incentive-compatible assignment rule weakly Pareto-dominates that provided by the dominant strategy equilibrium of DA.

(2001) probabilistic serial mechanism when the market is large (Che and Kojima, 2010). These mechanisms are dominated by either PM or BM, not only in the utilitarian sense, but also in the Rawlsian max-min sense. That is, PM and BM provide not only an efficient solution but also a fair solution to the problem we analyze.

Assignment problems are difficult to treat from the point of view of optimal mechanism design. The main difficulty arises from the multidimensionality of agents' valuations, which renders the problem practically unsolvable.⁸ In this line, Budish (2012) argues that the search for good properties using matching theory tools should be preferred to the search for an optimal solution. Despite agreeing with this, we have tried to analyze a tractable model so that we could obtain a first glance at what optimal mechanism design can tell us about assignment problems. What we found is very surprising: the optimal solution turns out to be well-known mechanisms in the matching literature under a wide range of parameters. Our findings enrich the debate, while we acknowledge the importance of other features: varying ordinal preferences for different classes,⁹ nonstrategic players, preexisting priorities, precedence, etc.

The rest of the paper is organized as follows. Section 2 introduces the model and the problem, for which we provide a solution. Section 3 analyzes the implementation of the optimal assignment rules. Section 4 discusses other assumptions of agents' preferences, other objective functions, and the failure of PM and BM being optimal. Section 5 concludes. An Appendix contains the longer proofs.

2 Allocation to Ranked Positions

There are three kind of positions: top, medium, and low. Every agent's utility from obtaining a top position is 1. Obtaining a low position gives zero utility. The medium position gives a utility of v , which is distributed over $[0, 1]$ according to continuous distribution $F \in \mathcal{C}^2$ and strictly positive density f . The valuation for the medium position is agents' private information. There is a continuum of agents with a mass 1, μ_1 measure of top positions, μ_2 measure of medium positions, and $1 - \mu_1 - \mu_2$ measure of low positions (of course, $\mu_1, \mu_2, 1 - \mu_1 - \mu_2 \in (0, 1)$).

⁸An example of how cumbersome the problem becomes is Armstrong (2000) for the design of optimal multi-object auctions.

⁹In subsection 4.1, we partially analyze a case in which the top 2 positions are not always ranked the same way for all agents and show that PM and BM can continue to be optimal under some reasonable assumptions.

We consider cardinal mechanisms without transfers, and by the revelation principle we focus on direct truthful mechanisms. An *assignment rule* is a function $(p, q) : [0, 1] \rightarrow \Delta$ where the range is the tridimensional simplex. In this notation, $p(v)$ and $q(v)$ denote the probability of being assigned to high and medium positions, respectively (since everybody will be allocated a position, the probability of being assigned to low position is $1 - p(v) - q(v)$).

We have the following market-clearing conditions:

$$\begin{aligned} \int_0^1 p(v) f(v) dv &= \mu_1, \\ \int_0^1 q(v) f(v) dv &= \mu_2, \end{aligned}$$

which need to be satisfied by any feasible rule (note that the third market clearing condition $\int_0^1 (1 - p(v) - q(v)) f(v) dv = 1 - \mu_1 - \mu_2$ is automatically satisfied). We say that a direct rule $(p(\cdot), q(\cdot))$ is *feasible* if it satisfies the above two market-clearing conditions, and $p(v), q(v), 1 - p(v) - q(v) \in [0, 1]$ for all $v \in [0, 1]$.

A rule is *truthful* (or *incentive compatible*) if no agent can be better off by announcing any other type. That is, for all $v, v' \in [0, 1]$, the following condition needs to hold:

$$p(v) + q(v)v \geq p(v') + q(v')v.$$

Define $U(v) = p(v) + q(v)v$. By the standard envelope theorem (or Myerson's) method, we first establish the following *payoff equivalence* result:

Lemma 1 *Incentive compatibility is equivalent to*

$$U(v) = p(0) + \int_0^v q(t) dt$$

and q increasing, p decreasing, $q + p$ increasing (together called monotonicity conditions).

Proof. The interim expected utility of a player with value v and announcement v' is

$$u(v, v') = p(v') + q(v')v$$

Incentive compatibility requires that $u(v, v')$ is maximized at $v = v'$, for which a necessary condition is the local incentive compatibility, which implies the derivative of $u(v, v')$ with

respect to v' at $v' = v$ to be zero:

$$p'(v) + q'(v)v = 0.$$

Since $U(v) = p(v) + q(v)v$ and $U'(v) = p'(v) + q'(v)v + q(v)$, we have the following necessary condition:

$$U'(v) = q(v).$$

By standard arguments, we can argue that $q(\cdot)$ has to be increasing and $p(\cdot)$ has to be decreasing. Moreover, it follows that $q(\cdot) + p(\cdot)$ has to be increasing. Also, again by standard arguments (see Krishna, 2002, Section 5.1.2), in this setup the “envelope condition” $U'(v) = q(v)$ together with monotonicity conditions (q increasing, p decreasing, $q + p$ increasing) is equivalent to incentive compatibility. Hence, we have

$$\begin{aligned} U(v) &= U(0) + \int_0^v q(t) dt \\ &= p(0) + \int_0^v q(t) dt \end{aligned}$$

which is equivalent to incentive compatibility. ■

Note that since p and q are the probability functions and given monotonicity conditions, any feasible rule has to satisfy the following conditions: $q(0) \geq 0$, $q(1) \leq 1$, $p(0) \leq 1$, $p(1) \geq 0$, $q(0) + p(0) \geq 0$, $q(1) + p(1) \leq 1$ (together called *boundary conditions*).

2.1 Utilitarian and Rawlsian Social Welfare

Utilitarian social welfare is given by

$$\begin{aligned} USW &= \int_0^1 U(v) f(v) dv \\ &= \int_0^1 (p(v) + q(v)v) f(v) dv \\ &= \int_0^1 p(v) f(v) dv + \int_0^1 q(v)v f(v) dv \\ &= \mu_1 + \int_0^1 q(v)v f(v) dv \end{aligned} \tag{1}$$

where the last line follows from the second of the market clearance conditions.

On the other hand, Rawlsian social welfare is equal to the lowest utility of the society. Since in any incentive-compatible rule the utility is increasing, Rawlsian social welfare is equal to the utility of 0 type, which is

$$RSW = p(0). \quad (2)$$

2.2 Achieving Highest Social Welfare

In this subsection, we further simplify the incentive compatibility and feasibility conditions by writing the constraints as a function of $q(\cdot)$ only. Before we move on to the simplification, we define virtual valuations and state our technical assumptions, which are fairly weak and standard.

Definition 2 Denote Myerson's (or buyers') virtual valuation by

$$J(t) = t - \frac{1 - F(t)}{f(t)}$$

and sellers' virtual valuation by

$$H(t) = t + \frac{F(t)}{f(t)}$$

Assumption We assume that H is increasing. This is a rather weak assumption; if F is logconcave,¹⁰ that is, if $\frac{F}{f}$ is nondecreasing, then H is obviously increasing. Many distributions widely used in the literature are logconcave: all power, normal, log-normal, Pareto, Weibull, Gamma, exponential, logistic, extreme value, Laplace, Chi distributions have logconcave CDF's, and thus also result in an increasing H . We also assume that J is increasing, which is a standard assumption in mechanism design literature and is also satisfied by many widely used distribution functions.

¹⁰For an excellent discussion on log-concave probability and its economic applications, see Bergstrom and Bagnoli (2005).

The proof of the following proposition is relegated to the Appendix. In the proof, first by Lemma (1) and changing the order of integration, we write $p(\cdot)$ as a function of $q(\cdot)$ and $F(\cdot)$. Second we show that $q(0) \geq 0$, $p(1) + q(1) \leq 1$ and $p(0) \geq 0$ implies the rest of the boundary constraints and rewrite these constraints as a function of $q(\cdot)$ and $F(\cdot)$ only. Finally, we show that q being increasing implies the other monotonicity constraints.

Proposition 3 *A rule $(p(\cdot), q(\cdot))$ is incentive compatible and feasible if and only if*

$$p(v) = \mu_1 + \int_0^1 J(t) q(t) f(t) dt + \int_0^v q(t) dt - q(v) v, \quad (3)$$

and

$$\begin{aligned} \int_0^1 q(t) f(t) dv &= \mu_2 \\ \int_0^1 H(t) q(t) f(t) dt &\leq 1 - \mu_1 \\ \int_0^1 H(t) q(t) f(t) dt &\geq q(1) - \mu_1 \\ q(0) &\geq 0 \\ q &\text{ is increasing} \end{aligned}$$

We ignore the last two constraints from now on, as they will be satisfied in the welfare-maximizing rules in what follows. We call the first constraint

$$\int_0^1 q(t) f(t) dt = \mu_2 \quad (4)$$

the market-clearing (*MC*) condition; the second inequality

$$\int_0^1 H(t) q(t) f(t) dt \leq 1 - \mu_1 \quad (5)$$

the upper bound for H (*UB*); and the third inequality

$$\int_0^1 H(t) q(t) f(t) dt \geq q(1) - \mu_1 \quad (6)$$

the lower bound for H (*LB*).

2.2.1 The Utilitarian Problem

By (1) and Proposition 3, and since μ_1 is a constant, utilitarian social welfare maximization is equivalent to

$$\max_{q(\cdot)} \int_0^1 q(t) t f(t) dt$$

subject to (4), (5), and (6).

2.2.2 The Rawlsian Problem

On the other hand, by (2), equation (29) in the proof of Proposition 3, Proposition 3, and since μ_1 is a constant, Rawlsian social welfare maximization is equivalent to

$$\max_{q(\cdot)} \int_0^1 J(t) f(t) q(t) dt$$

subject to (4), (5), and (6). To give some intuition behind this objective function, observe that the min-utility type is always the lowest type. This type does not value middle-class positions; therefore, a way to maximize her utility is to allow her to “sell” those to the other agents in exchange of probabilities for high-class positions. As a consequence, the objective function coincides with the objective function in revenue-maximizing auction design problems.

Next, we consider the solution of the two problems in which we ignore UB and LB constraints. The solution to this relaxed problem turns out to be useful for the solution of the original problem.

2.3 The Relaxed Problem

Let us consider the (utilitarian or Rawlsian) problem in which we consider only the first constraint, MC ($\int_0^1 q(t) f(t) dt = \mu_2$). It is easy to show the following:

Lemma 4 *In the relaxed problem, the solution to both the utilitarian and the Rawlsian problem is given by*

$$q^{R*}(t) = \begin{cases} 1 & t \in [v^*, 1] \\ 0 & t \in [0, v^*) \end{cases}$$

for v^* that satisfies

$$1 - F(v^*) = \mu_2$$

or

$$v^* = F^{-1}(1 - \mu_2)$$

Proof. This is simply because marginal benefit of increasing $q(t)$ is $tf(t)$ in the utilitarian problem and $J(t)f(t)$ in the Rawlsian problem. In both problems, the marginal cost is $f(t)$. The benefit-cost ratios are therefore given by t and $J(t)$, respectively. Since they are both increasing functions of t , in the optimal solution, we should not allocate any positive q 's to any t 's unless all higher $\hat{t}$'s are already maxed out at $q(\hat{t}) = 1$. Hence, the result obtains.

■

In what follows, we simplify $\int_0^1 H(t)q(t)f(t)dt$ at this optimal solution as

$$\begin{aligned} \int_{v^*}^1 H(t)f(t)dt &= \int_{v^*}^1 (tf(t) + F(t))dt \\ &= F(t)t \Big|_{t=v^*}^{t=1} \\ &= 1 - v^*F(v^*) \\ &= 1 - (1 - \mu_2)F^{-1}(1 - \mu_2), \end{aligned}$$

compare this optimal solution with $1 - \mu_1$, and analyze the different cases separately.

If $\int_0^1 H(t)f(t)dt$ for the optimal solution of the relaxed problem also satisfies UB, that is, if

$$1 - (1 - \mu_2)F^{-1}(1 - \mu_2) \leq 1 - \mu_1,$$

or

$$\mu_1 - (1 - \mu_2)F^{-1}(1 - \mu_2) \leq 0$$

we call this a *tough* economy (and otherwise a *mild* economy).¹¹ This is because $\mu_1 - (1 - \mu_2)F^{-1}(1 - \mu_2)$ is increasing in both μ_1 and μ_2 . Hence, lower capacities of high and medium hierarchical positions (i.e. a market that is competitive) would make this condition hold, whereas higher capacities of high and medium positions (i.e. a market that is not competitive) would violate this condition. Whether $\mu_1 - (1 - \mu_2)F^{-1}(1 - \mu_2)$ is greater or smaller than 0 turns out to be crucial in our model. We therefore formally introduce the following definitions.

¹¹Note that if the inequality holds as an equality, then both UB and LB are satisfied since $q(1) = 1$.

Definition 5 We call an economy with $\mu_1 - (1 - \mu_2) F^{-1}(1 - \mu_2) \leq 0$ a tough economy, and an economy with $\mu_1 - (1 - \mu_2) F^{-1}(1 - \mu_2) > 0$ a mild economy.

In the next section, we show that under a tough economy, the optimal solution has to be a particular two-step function that satisfies $q(0) = 0$ and LB has to hold. Then we show that under a mild economy, the optimal solution depends on the curvature of F .

2.4 Tough Economy

In a tough economy ($\mu_1 - (1 - \mu_2) F^{-1}(1 - \mu_2) \leq 0$), the utilitarian and Rawlsian optimal mechanism turn out to be identical to each other.

Proposition 6 In a tough economy, the optimal solution for both utilitarian and Rawlsian social welfare is

$$q^{T*}(t) = \begin{cases} \frac{\mu_1}{F(v^{**})v^{**}} & t \in [v^{**}, 1] \\ 0 & t \in [0, v^{**}) \end{cases} \quad (7)$$

for v^{**} that uniquely solves

$$\frac{F(v)v}{1 - F(v)} = \frac{\mu_1}{\mu_2}. \quad (8)$$

Proof. We know that $\mu_1 - (1 - \mu_2) F^{-1}(1 - \mu_2) \leq 0$. If $(1 - \mu_2) F^{-1}(1 - \mu_2) = \mu_1$, since

$$q^*(t) = \begin{cases} 1 & t \in [F^{-1}(1 - \mu_2), 1] \\ 0 & t \in [0, F^{-1}(1 - \mu_2)) \end{cases}$$

solves the problem subject to only MC and also satisfies UB and LB (as equalities since $\int_0^1 H(t) q(t) f(t) dt = 1 - \mu_1 = q(1) - \mu_1$ for q^*), hence it is the solution to the social welfare maximization problem for both the utilitarian and Rawlsian cases. Note that for $(1 - \mu_2) F^{-1}(1 - \mu_2) = \mu_1$, $v^* = F^{-1}(1 - \mu_2)$ is the unique solution to $\frac{F(v)v}{1 - F(v)} = \frac{\mu_1}{\mu_2}$. Hence, $v^* = v^{**}$, and the optimal solution satisfies (7).

Next, if $\mu_1 < (1 - \mu_2) F^{-1}(1 - \mu_2)$, then we can argue that UB never binds, since given that MC holds (and the assumption that H is increasing), the highest $\int_0^1 H(t) q(t) f(t) dt$ can get is $\int_{v^*}^1 H(t) f(t) dt$, which is by assumption smaller than $1 - \mu_1$.

We first establish the following lemma, whose proof is relegated to the Appendix. In the proof, we first argue that the optimal solution has to be a “two-step” function by showing that an “ironed out” version of a non-two-step function gets a better welfare. Then we show that the lower step has to be 0.

Lemma 7 *The optimal solution for both social welfare problems has to be of the form*

$$q(t) = \begin{cases} q(1) & t \in [d, 1] \\ 0 & t \in [0, d) \end{cases}$$

By Lemma 7, the maximization problem becomes

$$\max_{d, q(1)} q(1) \int_d^1 t f(t) dt$$

for the utilitarian maximization, and

$$\max_{d, q(1)} q(1) \int_d^1 J(t) f(t) dy$$

for the Rawlsian maximization. They are both subject to the following two constraints.

$$\begin{aligned} q(1)(1 - F(d)) &= \mu_2 \\ q(1) \int_d^1 H(t) f(t) dt &\geq q(1) - \mu_1. \end{aligned}$$

Note that the second equality can be written as

$$q(1)(1 - F(d)d) \geq q(1) - \mu_1$$

or

$$\mu_1 \geq q(1) F(d) d.$$

From the first constraint, we have

$$q(1) = \frac{\mu_2}{(1 - F(d))},$$

which can be incorporated to the objective functions, making them

$$\mu_2 \mathbb{E}[X | X \geq d]$$

for the utilitarian maximization, and

$$\mu_2 \mathbb{E} [J(X) | X \geq d]$$

for the Rawlsian maximization. They are both subject to

$$\mu_1 \geq \frac{\mu_2}{(1 - F(d))} F(d) d.$$

Furthermore, note that both objective functions are increasing in d and the right-hand side of the constraint is increasing in d . Therefore, optimal v^{**} uniquely solves

$$\frac{F(d) d}{(1 - F(d))} = \frac{\mu_1}{\mu_2}$$

and optimal $q(1)^*$ satisfies

$$q(1)^* = \frac{\mu_2}{(1 - F(v^{**}))} = \frac{\mu_1}{F(v^{**}) v^{**}}.$$

■

2.5 Mild Economy

In a mild economy ($\mu_1 - (1 - \mu_2) F^{-1}(1 - \mu_2) > 0$), it turns out that optimal rules can take different forms depending on the convexity or concavity of H and the monotonicity of $v/F(v)$.

Proposition 8 *In a mild economy, for the utilitarian maximization, we have:*

- (1) *if H is linear, we have infinitely many optimal rules.*
- (2) *if H is strictly convex, the optimal rule is¹²*

$$q^{M*}(t) = \begin{cases} \frac{\mu_2}{1 - F(d^*)} & t \in [d^*, 1] \\ 0 & t \in [0, d^*) \end{cases} \quad (9)$$

for $d^ \in (0, 1)$ that uniquely solves*

$$(1 - \mu_1 - d^* \mu_2) F(d^*) = 1 - \mu_1 - \mu_2. \quad (10)$$

¹²If H is convex, the given optimal mechanism is an optimal mechanism.

(3) if H is strictly concave, the optimal rule is¹³

$$q^{M^{**}}(t) = \begin{cases} 1 & t \in [d^{**}, 1] \\ 1 - \frac{1-\mu_2}{F(d^{**})} & t \in [0, d^{**}) \end{cases} \quad (11)$$

for

$$d^{**} = \frac{\mu_1}{1 - \mu_2} \quad (12)$$

As for Rawlsian maximization, if H is convex and $v/F(v)$ is decreasing, then q^{M^*} is optimal. If H is concave and $v/F(v)$ is increasing, then $q^{M^{**}}$ is optimal.

Proof. In the proof, we first show that UB has to bind at the optimal solutions, then solve for the maximizers among the two-step rules, then finally show that the two-step maximizer gives a higher social welfare than any q for which UB binds.

We first establish the following lemma, whose proof is relegated to the Appendix. In the proof, we first argue that if UB does not bind, then q cannot be optimal unless it is a two-step function (again by considering an “ironed out” version of the function); then we argue that the maximizer among the two-step functions has to satisfy UB.

Lemma 9 *UB has to bind at the optimal solution for both problems.*

Hence, without loss of generality we consider assignment rules such that UB binds. We first consider the Utilitarian problem.

Utilitarian Problem:

Let us solve for the maximizers among the two-step functions. That is, among the functions of the form

$$q(t) = \begin{cases} k & t \in [c, 1] \\ l & t \in [0, c) \end{cases}$$

for $c \in [0, 1]$, $0 \leq l < k \leq 1$.

Note that MC can be simplified as

$$lF(c) + k(1 - F(c)) = \mu_2 \quad (13)$$

¹³If H is concave, the given optimal mechanism is an optimal mechanism.

and that

$$\int_0^1 H(t) q(t) dt = lcF(c) + k(1 - cF(c)).$$

Also, we have to have

$$lcF(c) + k(1 - cF(c)) = 1 - \mu_1 \quad (14)$$

by Lemma 9.

In a two-step rule with parameters k, l, c where both MC and UB bind, we solve the equations (13) and (14) together for k and l and get

$$\begin{aligned} k(c) &= \frac{1 - \mu_1 - c\mu_2}{1 - c} \\ l(c) &= \frac{1}{F(c)(1 - c)} (F(c) + \mu_2 + \mu_1 - F(c)\mu_1 - F(c)c\mu_2 - 1) \\ &= \frac{1 - \mu_1 - c\mu_2}{1 - c} - \frac{1 - \mu_1 - \mu_2}{F(c)(1 - c)} \\ &= k(c) - \frac{1 - \mu_1 - \mu_2}{F(c)(1 - c)} \\ &\equiv k(c) - \frac{n(c)}{F(c)} \end{aligned}$$

$k(c) \leq 1$ implies

$$1 - c \geq 1 - \mu_1 - c\mu_2$$

or

$$c \leq \frac{\mu_1}{1 - \mu_2}$$

$l(c) \geq 0$ implies

$$(1 - \mu_1 - c\mu_2) F(c) \geq 1 - \mu_1 - \mu_2 \quad (15)$$

Notice that c is then acceptable if it lies on the interval $[d^*, \frac{\mu_1}{1 - \mu_2}]$, where d^* uniquely solves¹⁴

$$(1 - \mu_1 - d^*\mu_2) F(d^*) = 1 - \mu_1 - \mu_2.$$

¹⁴Existence: Take the function $G(d) = (1 - \mu_1 - d\mu_2)F(d) - (1 - \mu_1 - \mu_2)$. $G(0) < 0$, and now observe that $G(\frac{\mu_1}{1 - \mu_2}) = (1 - \mu_1 - \frac{\mu_1\mu_2}{1 - \mu_2})F(\frac{\mu_1}{1 - \mu_2}) - (1 - \mu_1 - \mu_2) > 0$. This is due to $F(\frac{\mu_1}{1 - \mu_2}) > 1 - \mu_2$ (as a condition of the proposition) and the fact that $1 - \mu_2 = \frac{1 - \mu_1 - \mu_2}{1 - \mu_1 - \frac{\mu_1\mu_2}{1 - \mu_2}}$. Since G is continuous, the intermediate value theorem applies. Uniqueness: $G'(d) = f(d)(1 - \mu_1 - d\mu_2) - \mu_2 F(d)$. Its sign is equivalent to that of $G'(d)/f(d) = 1 - \mu_1 - \mu_2 H(d)$ which is a decreasing function (since $H(\cdot)$ is increasing). Thus G has a unique local maximum, if any. Moreover, $G(1) = 0$, thus there is only one $d^* \in (0, 1)$ meeting $G(d^*) = 0$.

Also notice that the lowest c gives the rule q^{M*} stated in the proposition, while the highest c provides q^{M**} .

Let us denote the objective function among the two-step functions with cutoff c by $o(c)$. Note that $o(c)$ is equivalent to

$$\begin{aligned}
& l(c) \int_0^c t f(t) dt + k(c) \int_c^1 t f(t) dt \\
&= k(c) \mathbb{E}[X] - n(c) \mathbb{E}[X | X \leq c] \\
&= \frac{1 - \mu_1 - c\mu_2}{1 - c} \mathbb{E}[X] - \frac{1 - \mu_1 - \mu_2}{1 - c} \mathbb{E}[X | X \leq c] \\
&= \left(\frac{1 - \mu_1 - \mu_2}{1 - c} + \mu_2 \right) \mathbb{E}[X] - \frac{1 - \mu_1 - \mu_2}{1 - c} \mathbb{E}[X | X \leq c] \\
&= \left(\frac{1 - \mu_1 - \mu_2}{1 - c} \right) [\mathbb{E}[X] - \mathbb{E}[X | X \leq c]] + \mu_2 \mathbb{E}[X] \\
&= (1 - \mu_1 - \mu_2) \frac{\mathbb{E}[X] - \mathbb{E}[X | X \leq c]}{1 - c} + \mu_2 \mathbb{E}[X]
\end{aligned}$$

Define $T(x)$ as

$$\begin{aligned}
T(x) &= \mathbb{E}[X | X \leq x] \\
&= x - \frac{1}{F(x)} \int_0^x F(t) dt
\end{aligned}$$

It is immediate that $o(c)$ is decreasing (increasing) if and only if

$$S(c) = \frac{T(1) - T(c)}{1 - c}$$

is decreasing (increasing). And S is decreasing (increasing) if T is concave (convex).

In the remainder of the proof, we analyze the three cases, where we assume H to be linear, strictly convex, or strictly concave.

(1) H is linear

In this case, $H(v) = hv$ (note that intercept has to be zero in order not to violate the constraint $F(0) = 0$). Then the binding UB constraint would be rewritten as $\int_0^1 f(v)q(v)v = \frac{1-\mu_1}{h}$. That is, the objective function must take value $\frac{1-\mu_1}{h}$. Any feasible rule q such that UB binds obtains this result. The set of such rules is not empty. For instance, any two-step rule with parameters $c \in [d^*, \frac{\mu_1}{1-\mu_2}]$, $k(c)$ and $l(c)$ readily obtains $\int_0^1 f(v)q(v)v = \frac{1-\mu_1}{h}$, by construction. Also, any convex combination of these two-step rules obtains the same result.

(2) H is strictly convex

We first establish the following lemma, whose proof is relegated to the Appendix.

Lemma 10 *If H is strictly convex, then T is strictly concave (hence S is strictly decreasing).*

Then, among the two-step rules, the one depicted in the proposition, q^{M*} , is optimal. In the rest of the proof we show that no other rule such that both MC and UB bind can obtain higher welfare.

Each rule q has an associated density function $r(v) = \frac{q(v)f(v)}{\mu_2}$. Indeed, MC implies $\int_0^1 r(v)dv = 1$. $R(v) = \int_0^v r(t)dt$ is the distribution function associated to q . The maximization problem could be rewritten as finding a distribution function R , as follows:

$$\max_R \mathbb{E}_R v \text{ s.t. } \mathbb{E}_R H(v) = \frac{1 - \mu_1}{\mu_2}.$$

The problem is incomplete because R should be compatible with a monotonic q . Yet for the moment this is irrelevant. We use the notation R^* and R^{**} for the distributions associated with q^{M*} and q^{M**} , respectively. Notice that $\mathbb{E}_{R^*} H(v) = \mathbb{E}_{R^{**}} H(v) = \frac{1 - \mu_1}{\mu_2}$.

Next, we establish the following lemma, whose proof is relegated to the Appendix.

Lemma 11 *If $E_R v = E_{R^*} v$ (or $= E_{R^{**}} v$), then R either second-order stochastically dominates, or is dominated by, R^* (R^{**}).*

Corollary 12 *An immediate implication of Lemma 11 and the assumption of a strictly convex H is that $E_R H(v) \neq \frac{1 - \mu_1}{\mu_2}$.*

Finally, we have the tools to show that q^{M*} is optimal. By the method of contradiction suppose that there is a feasible q and its associated distribution R is such that $\mathbb{E}_R v > \mathbb{E}_{R^*} v$. Since q^{M**} is strictly worse than q^{M*} , we have $\mathbb{E}_{R^{**}} v < \mathbb{E}_{R^*} v$. Then, for some $\lambda \in (0, 1)$, we can find $q^\lambda = \lambda q + (1 - \lambda)q^{M**}$ ¹⁵ with associated distribution $R^\lambda = \lambda R + (1 - \lambda)R^{**}$ such that $\mathbb{E}_{R^\lambda} v = \mathbb{E}_{R^*} v$. From the previous corollary we know that $\mathbb{E}_{R^\lambda} H(v) \neq \mathbb{E}_{R^*} H(v) = \frac{1 - \mu_1}{\mu_2}$. But since $\mathbb{E}_{R^{**}} H(v) = \frac{1 - \mu_1}{\mu_2}$ and $\mathbb{E}_{R^\lambda} H(v) = \lambda \mathbb{E}_R H(v) + (1 - \lambda) \mathbb{E}_{R^{**}} H(v)$, we obtain $\mathbb{E}_R H(v) \neq \frac{1 - \mu_1}{\mu_2}$, which is a contradiction.

¹⁵Note that any convex combination of two feasible mechanisms is also feasible.

(3) H is strictly concave

The proof is analogous to the case where H is strictly concave, with analogous claims. In this case, we argue that if H is strictly concave, then T is strictly convex (hence S is strictly increasing). Thus, q^{M**} is optimal among two-step rules. To show that no other feasible rule q obtains higher welfare, the analogous results apply. By method of contradiction, consider a feasible q and its associated distribution R be such that $\mathbb{E}_R v > \mathbb{E}_{R^{**}} v$. Since q^{M**} is strictly better than q^{M*} , we have $\mathbb{E}_{R^{**}} v > \mathbb{E}_{R^*} v$. For some $\lambda \in (0, 1)$, we can define $q^\lambda = \lambda q + (1 - \lambda)q^*$ with associated distribution $R^\lambda = \lambda R + (1 - \lambda)R^*$ such that $\mathbb{E}_{R^\lambda} v = \mathbb{E}_{R^{**}} v$. We have that R^λ either second-order dominates, or is dominated by, R^{**} . Hence we know that $\mathbb{E}_{R^\lambda} H(v) \neq \mathbb{E}_{R^{**}} H(v) = \frac{1 - \mu_1}{\mu_2}$. But since $\mathbb{E}_{R^*} H(v) = \frac{1 - \mu_1}{\mu_2}$ and $\mathbb{E}_{R^\lambda} H(v) = \lambda \mathbb{E}_R H(v) + (1 - \lambda) \mathbb{E}_{R^*} H(v)$, we obtain $\mathbb{E}_R H(v) \neq \frac{1 - \mu_1}{\mu_2}$, which is a contradiction.

Next, we consider the Rawlsian maximization problem.

Rawlsian Problem:

Solving for the maximizers among the two-step functions as in the utilitarian case, we have

$$\begin{aligned} k(c) &= \frac{1 - \mu_1 - c\mu_2}{1 - c} \\ l(c) &= k(c) - \frac{1 - \mu_1 - \mu_2}{F(c)(1 - c)} \end{aligned}$$

and c is acceptable if it lies on the interval $[d^*, \frac{\mu_1}{1 - \mu_2}]$, where d^* uniquely solves

$$(1 - \mu_1 - d^* \mu_2) F(d^*) = 1 - \mu_1 - \mu_2.$$

Let us denote the objective function among the two-step functions with cutoff c by $ro(c)$. We can then see that $ro(c)$ is equivalent to (note that $\int J(t) f(t) dt = -t(1 - F(t))$)

$$\begin{aligned} l(c) \int_0^c J(t) f(t) dt + k(c) \int_c^1 J(t) f(t) dt \\ &= l(c)(-c(1 - F(c))) + k(c)(c(1 - F(c))) \\ &= c(1 - F(c))(k(c) - l(c)) \\ &= (1 - \mu_1 - \mu_2) \frac{c(1 - F(c))}{F(c)(1 - c)}. \end{aligned}$$

Hence, the optimal c among the two-step allocations is given by

$$c^* = \arg \max_{c \in [d^*, \frac{\mu_1}{1-\mu_2}]} \tilde{S}(c) = \frac{c(1-F(c))}{F(c)(1-c)}.$$

Denoting $\tilde{T}(c) = \mathbb{E}(J(v)|v \leq c) = -c \frac{1-F(c)}{F(c)}$ and noticing that $\tilde{T}(1) = 0$, we have that $\tilde{S}(c) = \frac{\tilde{T}(1)-\tilde{T}(c)}{1-c}$.

Now, we establish the following lemma, whose proof is relegated to the Appendix.

Lemma 13 (1) *If H is strictly convex and $v/F(v)$ is decreasing, then $\tilde{T}$ is strictly concave (hence $\tilde{S}$ is strictly decreasing).*

(2) *If H is strictly concave and $v/F(v)$ is increasing, then $\tilde{T}$ is strictly convex (hence $\tilde{S}$ is strictly increasing).*

In case (1), the Rawlsian-optimal two-step rule implies $c = d^*$. In case (2), the Rawlsian-optimal two-step rule implies $c = \frac{\mu_1}{1-\mu_2}$. In both cases, as in the utilitarian problem, no other feasible assignment rule such that UB binds can obtain a better value for the objective function. The argument for this step is identical to the one in the utilitarian case, and therefore we omit it here. ■

3 Implementation via pseudomarket and Boston Mechanism

In this section, we solve for equilibria of Hylland and Zeckhauser (1979)'s pseudomarket with equal budgets (PM) and Boston mechanism without priorities (BM) in our setup. We establish that, in a tough market, they both result in the same allocation as the optimal allocations in the utilitarian and Rawlsian setups. We also show that in a mild market, the competitive equilibrium of PM implements q^{M*} , and the Bayesian Nash equilibrium of BM implements q^{M**} . Hence when H is strictly convex (concave), PM (BM) implements the utilitarian optimal assignment. Moreover, when H is strictly convex and $v/F(v)$ is decreasing (H is strictly concave and $v/F(v)$ is increasing), PM (BM) implements the Rawlsian optimal assignment.

We first formally describe and analyze the equilibria of the two mechanisms. We also show the uniqueness of the equilibria. We finally compare the equilibrium assignments to those arising from the optimal mechanisms.

3.1 Pseudomarket with Equal Budgets

In Hylland and Zeckhauser (1979)'s pseudomarket with equal budgets (*PM*), all agents are endowed with the same budget, which is normalized to 1. This budget consists of “fake money,” and thus does not generate utility per se. It is used to purchase probability units of the available social positions. We normalize the price for low-class positions to 0, and we denote the price for upper-class positions as π_1 and the price for medium-class positions as π_2 . Without loss of generality, we set $\pi_1 > \pi_2 > 0$. Also, we can argue that $\pi_1 \geq 1$. This is because any other type of price vector would entitle all agents to get the high position and therefore could not possibly clear the market. Let $p(v, \pi_1, \pi_2)$ be the probability that a v -type agent buys to in order to be assigned the upper-class position. Also, let $q(v, \pi_1, \pi_2)$ denote her purchased probability of a medium-class position. A v -type agent solves the problem

$$\max_{p,q} p + qv \quad s.t. \pi_1 p + \pi_2 q \leq 1 \quad (16)$$

where $p, q \geq 0$, $p + q \leq 1$. The solution to the problem is unique for almost all types $v \in [0, 1]$,¹⁶ and it is denoted as $p^*(v, \pi_1, \pi_2)$ and $q^*(v, \pi_1, \pi_2)$.

A competitive equilibrium (*CE*) is a price vector $(\pi_1^*, \pi_2^*, 0)$ that satisfies (i) $\int_0^1 p^*(v, \pi_1^*, \pi_2^*) f(v) dv = \mu_1$ and (ii) $\int_0^1 q^*(v, \pi_1^*, \pi_2^*) f(v) dv = \mu_2$. An associated CE assignment is composed of the functions $p^*(\cdot, \pi_1^*, \pi_2^*)$ and $q^*(\cdot, \pi_1^*, \pi_2^*)$.

Existence of CE is guaranteed from Hylland and Zeckhauser (1979). The following proposition solves for the CE of the PM, and it also establishes *uniqueness* of CE. The proof is relegated to the Appendix. In the proof, we analyze two cases: the first is when medium positions are expensive ($\pi_2^* \geq 1$), and the second is when medium positions are cheap ($\pi_2^* < 1$). It turns out that the former case corresponds to a tough market, and the latter one corresponds to a mild market.

¹⁶For all the types except for $v = \frac{\pi_1}{\pi_2}$.

Proposition 14 *In a tough market, the unique CE prices are*

$$\begin{aligned}\pi_1^* &= \frac{F(v^*)}{\mu_1} \\ \pi_2^* &= \frac{1 - F(v^*)}{\mu_2}\end{aligned}$$

where v^* solves

$$\frac{F(v^*)v^*}{1 - F(v^*)} = \frac{\mu_1}{\mu_2}. \quad (17)$$

In this CE, types $v < v^*$ obtain the probability bundle

$$\left(\frac{\mu_1}{F(v^*)}, 0, 1 - \frac{\mu_1}{F(v^*)}\right), \quad (18)$$

and types $v > v^*$ get

$$\left(0, \frac{\mu_2}{1 - F(v^*)}, 1 - \frac{\mu_2}{1 - F(v^*)}\right). \quad (19)$$

(where the first, second, and third components represent probability shares of high, medium, and low positions, respectively).

On the other hand, in a mild market, the unique CE prices are

$$\begin{aligned}\pi_1^* &= \frac{F(v^*)}{F(v^*) - (1 - \mu_1 - \mu_2)} \\ \pi_2^* &= \frac{1}{\mu_2} \frac{(1 - \mu_1)F(v^*) - (1 - \mu_1 - \mu_2)}{F(v^*) - (1 - \mu_1 - \mu_2)}\end{aligned}$$

where v^* solves

$$(1 - \mu_1 - v^*\mu_2) F(v^*) = 1 - \mu_1 - \mu_2. \quad (20)$$

In this CE, the assignment of probabilities is as follows. Types $v < v^*$ obtain the probability bundle

$$\left(1 - \frac{1 - \mu_1 - \mu_2}{F(v^*)}, 0, \frac{1 - \mu_1 - \mu_2}{F(v^*)}\right), \quad (21)$$

and types $v > v^*$ get

$$\left(1 - \frac{\mu_2}{1 - F(v^*)}, \frac{\mu_2}{1 - F(v^*)}, 0\right). \quad (22)$$

3.2 Boston Mechanism without Priorities

In the Boston mechanism without priorities (*BM*), agents simultaneously rank social classes. A round-by-round algorithm serves to assign the available positions. In a first round, we consider each agent for the social class she ranked in the top position. When there is an excess of considered individuals with respect to available positions, a fair lottery defines who is accepted and who is rejected. If there is no such excess demand, every considered agent is accepted. An accepted agent obtains a position for which he was being considered. The rejected agents go to a second round in which they are considered for the social class they ranked second, and a similar acceptance/rejection procedure applies for the not-yet-assigned positions. After a finite number of rounds (three at most in this case), all agents are eventually accepted for some social class position. It is apparent that no agent would rank the low-class position other than last. So the strategy space simplifies to two relevant strategies: either ranking the high class in first position (strategy 1) or ranking the middle class in first position (strategy 2). Let m_j be the mass of agents using strategy $j \in \{1, 2\}$.

In Proposition 15, we solve for the *unique* Bayesian Nash equilibrium of BM. The proof is relegated to the Appendix. In the proof, as in the proof of Proposition 14, we analyze two cases: first when medium positions are overdemanded ($m_2 \geq \mu_2$) and second when medium positions are underdemanded ($m_2 < \mu_2$). It turns out that the former case corresponds to a tough market, and the latter corresponds to a mild market.

Proposition 15 *In the unique Bayesian Nash equilibrium of BM, agents with values greater than v^* use strategy 2, and agents with values smaller than v^* use strategy 1. In a tough market, v^* is the solution to*

$$\frac{F(v^*)v^*}{1 - F(v^*)} = \frac{\mu_1}{\mu_2}, \quad (23)$$

whereas in a mild market,

$$v^* = \frac{\mu_1}{1 - \mu_2}. \quad (24)$$

In the bayesian Nash equilibrium of the tough market, types $v < v^$ obtain the probability bundle*

$$\left(\frac{\mu_1}{F(v^*)}, 0, 1 - \frac{\mu_1}{F(v^*)}\right), \quad (25)$$

whereas types $v > v^$ get*

$$\left(0, \frac{\mu_2}{1 - F(v^*)}, 1 - \frac{\mu_2}{1 - F(v^*)}\right). \quad (26)$$

On the other hand, in the Bayesian Nash equilibrium of the mild market, types $v < v^$ obtain*

the probability bundle

$$\left(\frac{\mu_1}{F(\frac{\mu_1}{1-\mu_2})}, 1 - \frac{1-\mu_2}{F(\frac{\mu_1}{1-\mu_2})}, \frac{1-\mu_1-\mu_2}{F(\frac{\mu_1}{1-\mu_2})}\right) \quad (27)$$

whereas types $v > v^*$ get

$$(0, 1, 0) \quad (28)$$

for types $v < \frac{\mu_1}{1-\mu_2}$.

3.3 Implementation of the Optimal Rules

As a corollary to Propositions 6, 8, 14, and 15, we establish our main result. The proof follows from noting that in a tough market both PM and BM result in allocation q^{T*} , and in a mild market PM results in allocation q^{M*} , whereas BM results in allocation q^{M**} .^{17,18}

Theorem 16 *In a tough market, utilitarian and Rawlsian optimal assignment is implemented by both the competitive equilibrium of the pseudomarket with equal budgets and the Nash equilibrium in the Boston mechanism without priorities. In a mild market, the utilitarian optimal assignment is implemented by competitive equilibrium of the pseudomarket if H is convex, and by Boston mechanism without priorities if H is concave.*

For the Rawlsian problem in a mild market, the optimal assignment rule is implemented by competitive equilibrium of the pseudomarket if H is convex and $c/F(c)$ is decreasing, and by Boston mechanism without priorities if H is concave and $c/F(c)$ is decreasing.

4 Discussion

In the following subsections we discuss several observations regarding the results of our model. First, we consider an extension of our model in which we allow for varying ordinal preferences of agents. Specifically, we suppose that high positions gives a constant utility $a \in (0, 1]$ to every agent. In this extension, we show that PM and BM may implement Utilitarian and

¹⁷More formally, it is achieved by noting the equality between, (i) v^{**} in (8) and v^* s in (17) and (23), (ii) q^{T*} in (7) and the second components of (18), (19), (25), and (26), (iii) d^* in (10) and v^* in (20), (iv) d^{**} in (12) and v^* in (24), (v) q^{M*} in (9) and the second components of (21), and (22), and finally (vi) q^{M**} in (11) and the second components of (27), and (28).

¹⁸Note that we only check for the probabilities of being allocated to the medium positions. This is sufficient as since the mechanisms are incentive compatible, payoff equivalence implies the probabilities of being allocated to high or low positions are the same as well.

Rawlsian maximization problems under some assumptions. Second, we illustrate that not all weighted utilitarian welfare functions have the same maximand in a tough market. Therefore, the coincidence between utilitarian and Rawlsian optimal rules in tough markets is not a subcase of a general equivalence across weighted welfare functions. Third, in a mild market, we show that the utilitarian- and the Rawlsian-optimal assignments may not coincide. For instance, it could be the case that BM provides the solution for one problem while PM provides the solution for the other problem. Finally, we show that neither the utilitarian- nor the Rawlsian-optimal assignments are always given by either PM or BM. With this we illustrate that the optimality of either PM or BM cannot be extended to every possible case.

4.1 Varying ordinal preferences

A natural question is whether our result collapse as we relax our assumption that high-class positions are the favorite ones for everyone. We work on the same model, but suppose that the high positions gives a constant utility $a \in (0, 1]$ to every agent. If a is high enough (we do not require a to be arbitrarily close to 1, we assume $a \geq H(F^{-1}(1 - \mu_2))$,¹⁹) that is, if high-class positions are still highly-valued, we find the following facts:²⁰

1. Tough markets emerge “more easily” than when $a = 1$, for fixed primitives F , μ_1 and μ_2 . Intuitively, lower a implies higher demand for mid-class positions, hence the likely the impossibility of ensuring mid-class positions to high types.
2. In tough markets, both BM and PM are still optimal, in the utilitarian as well as in the Rawlsian sense.
3. In mild markets, a bias in favor of BM arises. Sufficient conditions for optimality of BM are still sufficient in this extended model. The intuition is similar to the one of the first stated fact. Being medium positions more appreciated overall, BM becomes more appealing than PM. The former guarantees sure assignment of mid positions to sufficiently high types, while the latter does not.

Define $U(v) = ap(v) + q(v)v$. We first establish the following *payoff equivalence* result. Its proof’s steps are the same as in proof of Lemma 1.

¹⁹It is possible to achieve similar results for the case $a < H(F^{-1}(\mu_2))$, yet the analyzed case is simpler to illustrate.

²⁰While we explain the model in the lines below, we do not provide a full statement and proof of these results due to its large extent. All of them are available upon request.

Lemma 17 *Incentive compatibility is equivalent to*

$$U(v) = ap(0) + \int_0^v q(t) dt$$

and q increasing, p decreasing, $q + p$ increasing for $v \in [0, a]$ and decreasing for $v \in [a, 1]$ (together called monotonicity conditions).

Note that the extended problem becomes more complex than the original one since UB applies to a instead of the highest type: we need to ensure that $p(a) + q(a) \leq 1$.

The utilitarian social welfare can be shown to be

$$USW = a\mu_1 + \int_0^1 q(v) v f(v) dv$$

The Rawlsian social welfare is equal to

$$RSW = ap(0)$$

We establish the following result. The proof is similar to proof of Proposition 3.

Proposition 18 *A rule $(p(\cdot), q(\cdot))$ is incentive compatible and feasible if and only if*

$$p(v) = \mu_1 + \frac{1}{a} \left(\int_0^1 J(t) q(t) f(t) dt + \int_0^v q(t) dt - q(v) v \right),$$

and

$$\begin{aligned} \int_0^1 q(t) f(t) dv &= \mu_2 \\ \int_0^a H(t) q(t) f(t) dt + \int_a^1 J(t) q(t) f(t) dt &\leq a - a\mu_1 \\ \int_0^1 H(t) q(t) f(t) dt &\geq q(1) - a\mu_1 \\ q(0) &\geq 0 \text{ and } q \text{ is increasing} \end{aligned}$$

The first constraint is market clearing (MC), the second one is the Upper Bound (UB) and the third one is the Lower bound (LB).

We can easily see that, as in the original model, Utilitarian social welfare maximization is equivalent to

$$\int_0^1 q(t) t f(t) dt$$

subject to above MC, UB, LB.

On the other hand, Rawlsian social welfare maximization is equivalent to

$$\int_0^1 J(t) q(t) f(t) dt$$

subject to MC, UB, LB.

In the relaxed problem, in which we only consider MC (and ignore UB and LB,) since the objective functions and MC are the same, Lemma 4 holds and the solution of the relaxed problem of both Utilitarian and Rawlsian problem is given by

$$q^*(t) = \begin{cases} 1 & t \in [v^*, 1] \\ 0 & t \in [0, v^*) \end{cases}$$

for $v^* = F^{-1}(1 - \mu_2)$.

Since we assume $a \geq H(F^{-1}(1 - \mu_2)) (> F^{-1}(1 - \mu_2))$, the optimal rule in the relaxed problem satisfies UB if and only if

$$\begin{aligned} a - a\mu_1 &\geq \int_{F^{-1}(1-\mu_2)}^a H(t) f(t) dt + \int_a^1 J(t) f(t) dt \\ &= a - (1 - \mu_2) F^{-1}(1 - \mu_2) \end{aligned}$$

or

$$a\mu_1 - (1 - \mu_2) F^{-1}(1 - \mu_2) \leq 0$$

As in our main model, we call an economy a *tough* market if this condition is satisfied. Otherwise it is considered a *mild* market. Note that in this extension, it is more likely that we have a tough market: the condition in the main model required $(1 - \mu_2) F^{-1}(1 - \mu_2)$ to be greater than μ_1 , in the extension it should only be greater than $a\mu_1$. This brings us the first fact we stated.

As for the second statement, we first observe that when $a \geq H(F^{-1}(1 - \mu_2))$, in a maximization problem where the objective function is $\int_0^a H(t) q(t) f(t) dt + \int_a^1 J(t) q(t) f(t) dt$ and the constraint is just MC, the solution is q^* . This is because, $E[J(t) | t \geq v] = v$ and therefore in the optimal solution, q has to be constant between $b \equiv H^{-1}(a)$ and 1. Given this, and also $b > F^{-1}(1 - \mu_2)$, one can easily argue that the highest value that LHS of the UB constraint can get is achieved by q^* , and this means that in a tough market UB never binds. With this simplifying insight, the proof of the following proposition follows the same steps as in the proof of Proposition 6, and it is skipped.

Proposition 19 *For $a \geq H(F^{-1}(1 - \mu_2))$, in a tough market, the optimal solution for both Utilitarian and Rawlsian social welfare is*

$$q^a(t) = \begin{cases} \frac{\mu_1}{F(v^a)v^a} & t \in [v^a, 1] \\ 0 & t \in [0, v^a) \end{cases}$$

for v^a that uniquely solves

$$\frac{F(v)v}{1 - F(v)} = \frac{a\mu_1}{\mu_2}.$$

The next main result follows immediately. It stems from similar arguments as in Proposition 14, Proposition 15, and Theorem 16.

Proposition 20 *For $a \geq H(F^{-1}(1 - \mu_2))$, in a tough market, the utilitarian and Rawlsian optimal assignment is implemented by both PM and BM.*

As for mild markets, we can get results in this extension similar to the ones in our main model. For mild markets, it turns out that under the same conditions as in Theorem 16, BM continue to be optimal in the utilitarian and Rawlsian problems. Yet, sufficient conditions for PM being optimal in Theorem 16 need to be strengthened.²¹ Hence, in a way, this extension brings a bias in favor of BM, rather than PM, as we stated at the beginning of the subsection.

Next, we demonstrate that in a tough market PM and BM may fail to maximize a weighted utilitarian social welfare maximization problem.

4.2 Weighted utilitarian maximization in a tough market

In this subsection, we consider a weighted utilitarian objective and demonstrate that even in a tough market, the optimal assignment for weighted utilitarian may differ from the Utilitarian maximizing, and therefore also different from PM and BM allocations.

²¹For two-step assignment rules with cut-off c and UB binding, in this extension it turns out that the objective function becomes $S(c)$ multiplied by a factor $(1 - c)/(a - c)$. This factor is increasing in c , and this brings the bias in favor of higher c .

In the weighted utilitarian maximization, consider the weight $\alpha(v)$ for type v with

$$\int_0^1 \alpha(v) f(v) dv = 1$$

denote $\alpha(v) f(v)$ by $g(v)$ and $\int_0^v g(t) dt$ by $G(v)$.

The weighted utilitarian social welfare is given by

$$\begin{aligned} WUSW &= \int_0^1 \alpha(v) U(v) f(v) dv \\ &= \int_0^1 \alpha(v) (p(v) + q(v)v) f(v) dv \end{aligned}$$

subject to

$$\begin{aligned} \int_0^1 q(t) f(t) dv &= \mu_2 \\ \int_0^1 H(t) q(t) f(t) dt &\leq 1 - \mu_1 \\ \int_0^1 H(t) q(t) f(t) dt &\geq q(1) - \mu_1 \end{aligned}$$

Since $p(v) = \mu_1 + \int_0^1 J(t) q(t) f(t) dt + \int_0^v q(t) dt - q(v)v$, we can write $WUSW$ as

$$WUSW = \int_0^1 \alpha(v) \left(\mu_1 + \int_0^1 J(t) q(t) f(t) dt + \int_0^v q(t) dt \right) f(v) dv$$

and since $\int_0^1 \alpha(v) f(v) dv = 1$, we have

$$WUSW = \mu_1 + \int_0^1 J(t) q(t) f(t) dt + \int_0^1 \left(\int_0^v q(t) dt \right) g(v) dv$$

Finally, by changing the order of integration in the third term, we can write²²

$$WUSW = \mu_1 + \int_0^1 \left(t + \frac{F(t) - G(t)}{f(t)} \right) f(t) q(t) dt$$

Define $t + \frac{F(t) - G(t)}{f(t)}$ to be $K(t)$, as long as $K(t)$ is increasing, for the “tough market,” we would have the same solution as the utilitarian or Rawlsian problem.

However, the following example shows that the solution to weighted utilitarian may differ from the solution to utilitarian and Rawlsian problems.

Example 21 Consider $F(t) = t$, $\alpha(t) = 3t^2$, (so that $f(t) = 1$, $G(t) = t^3$, $H(t) = 2t$), $\mu_2 = 0.2$, and $\mu_1 = 0.64$. For these specifications, $K(t) = 2t - t^3$.

Then, the relaxed problem in weighted utilitarian case is to maximize

$$\int_0^1 (2t - t^3) q(t) dt$$

subject to

$$\int_0^1 q(t) dt = 0.2$$

The solution to this problem is given by the solution to the following problem

$$\max_c \frac{0.1}{1 - c} \int_c^1 (2t - t^3) dt$$

which has the solution of

$$c = 0.72076$$

Hence, the optimal solution for the relaxed problem of weighted utilitarian case is

$$q^*(t) = \begin{cases} 0.71623 & t \geq 0.72076 \\ 0 & t < 0.72076 \end{cases}$$

²²This can be achieved by noting the following.

$$\begin{aligned} WUSW &= \mu_1 + \int_0^1 J(t) q(t) f(t) dt + \int_0^1 q(t) (1 - G(t)) dt \\ &= \mu_1 + \int_0^1 (tf(t) - 1 + F(t) + 1 - G(t)) q(t) dt \\ &= \mu_1 + \int_0^1 \left(t + \frac{F(t) - G(t)}{f(t)} \right) f(t) q(t) dt \end{aligned}$$

Moreover, since

$$0.71623 \int_{0.72076}^1 2tdt = 0.34415$$

we can confirm that this solution also satisfies UB and LB ($0.34415 \leq 1 - 0.64$ and $0.34415 \geq 0.71623 - 0.64$), this is the solution to the original problem as well.

Finally, for these specifications, we can quickly check that the optimal solution for the utilitarian and Rawlsian case is different from above solution. It is

$$q^*(t) = \begin{cases} 1 & t \geq 0.8 \\ 0 & t < 0.2 \end{cases}$$

Next, we consider an example that shows that in a mild market, the Utilitarian and Rawlsian optimal assignments may differ, but coincide with either PM or BM.

4.3 Differing Utilitarian- and Rawlsian-optimal assignments in a mild market

A first glance at the sufficient conditions we find for the optimality of either the PM assignment rule or the BM assignment rule²³ in a mild market reveals that convexity/concavity of H is a sufficient condition in the utilitarian problem, whereas this alone does not apparently suffice in the Rawlsian problem. Some additional condition on the strength of F relative to the uniform distribution was needed in the proof. A fair question is whether this is a meaningful difference or we simply were unable to find the right proof in the Rawlsian problem. It turns out that there is a meaning difference that strength of F relative to the uniform distribution makes. The following example illustrates that the strength of F has an importance that can indeed counterbalance that of the convexity/concavity of H .

Example 22 Consider $F(v) = \frac{v+v^2}{2}$. A rapid calculation reveals that $H(v) = \frac{v(3v+2)}{2v+1}$ is a strictly concave function, therefore the (incentive-compatible) utilitarian-optimal assignment rule in a mild market is the BM assignment rule. However, observe that $\frac{v}{F(v)} = \frac{2v}{v+v^2}$ is a decreasing function, thus our results do not tell us what would the Rawlsian-optimal assignment rule be. In our sufficient conditions (Proposition 8), either H is concave and $v/F(v)$

²³In a mild market, the PM assignment rule is the binding-UB two-step assignment rule with the lowest possible cut-off c . By BM assignment rule we mean the binding-UB two-step assignment rule with the highest possible cut-off c .

is increasing or H is convex and $v/F(v)$ is decreasing. In this example, these two elements do not push towards the same direction.

It turns out that we can easily calculate $\tilde{T}(c) = \mathbb{E}(J(v)|v \leq c) = c - \frac{1}{1/2+c/2}$. We observe that $\tilde{T}$ is a concave function, therefore the Rawlsian-optimal two-step assignment rule in a mild market is the one provided by PM. By strict concavity of H , no other assignment rule with binding UB could improve the objective function. We conclude that the PM assignment rule is (incentive-compatible) Rawlsian-optimal.²⁴ Observe the contrast with BM providing the utilitarian-optimal assignment rule.

Next, we consider an explicit example that shows that in a mild market, the Utilitarian and Rawlsian optimal assignments may differ from the allocations of PM and BM.

4.4 Failure of PM and BM being optimal in a mild market

In Section 2.5, we give sufficient conditions for PM or BM to be optimal. Now, we provide an explicit example in which neither PM, nor BM is optimal in the utilitarian and Rawlsian maximization problems. In order to achieve this we consider an F such that H is neither convex, nor concave.

In the next example, the utilitarian and Rawlsian optimal mechanism is different from both PM and BM allocations.

Example 23 Consider

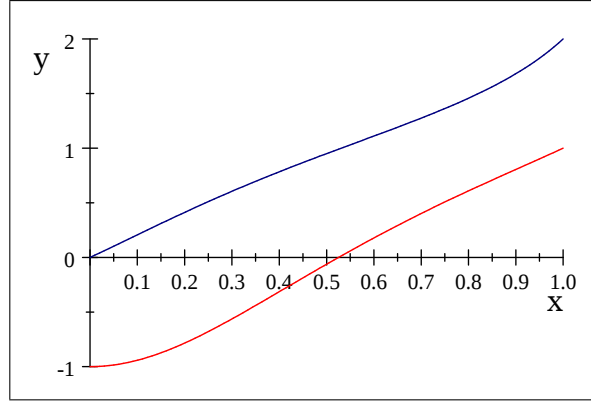
$$F(x) = \frac{x}{x^3 - 2x^2 + x + 1}$$

then can calculate that

$$\begin{aligned} H(x) &= \frac{x(-x^3 + x + 2)}{-2x^3 + 2x^2 + 1} \\ J(x) &= \frac{-x^6 + 4x^5 - 7x^4 + 2x^3 + 4x^2 - 1}{-2x^3 + 2x^2 + 1} \end{aligned}$$

²⁴The proof of these assertions follow the lines of the proof of Proposition 8.

and they are both increasing: (where H is the blue line and J is the red line)

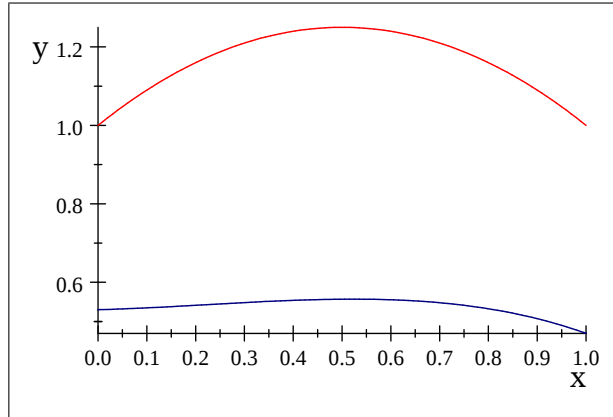


Moreover, we have

$$S(x) = \frac{1 - \int_0^1 \frac{t}{t^3 - 2t^2 + t + 1} dt - \left(x - \frac{\int_0^x \frac{t}{t^3 - 2t^2 + t + 1} dt}{\frac{x}{x^3 - 2x^2 + x + 1}} \right)}{1 - x}$$

$$\tilde{S}(x) = -x^2 + x + 1$$

and we can see that they both have inverse U shapes: (where S is the blue line and $\tilde{S}$ is the red line.)



We can calculate that S is maximized at 0.5 and $\tilde{S}$ is maximized numerically at 0.523.

Now, consider μ_1 and μ_2 such that $d^* < 0.5$ and $0.523 < d^{**}$ where $d^{**} = \frac{\mu_1}{1 - \mu_2}$ and d^* solves $(1 - \mu_1 - d^* \mu_2) F(d^*) = 1 - \mu_1 - \mu_2$.

Specifically, $\mu_1 = 0.3$, and $\mu_2 = 0.6$, then we have $d^{**} = \frac{0.3}{0.4} = 0.75$, and d^* solves $(0.7 - 0.6x) \frac{x}{x^3 - 2x^2 + x + 1} - 0.1 = 0$, with numerical solution $d^* = 0.19258$.

For this example, neither the PM allocation nor the BM allocation coincide with either the Utilitarian optimal allocation or Rawlsian optimal allocations are all different from each other. Specifically, PM allocation is a two-step function with cutoff approximately at $d^ = 0.19258$, BM allocation is a two-step function with cutoff at $d^{**} = 0.75$, Utilitarian optimal allocation is a two-step function with cutoff approximately at 0.523, and finally Rawlsian optimal allocation is a two-step function with cutoff at 0.5.*

5 Conclusion

Except for Miralles (2012), the use of optimal mechanism design tools for the study of assignment problems are virtually nonexistent. In this paper we consider a tractable problem with a continuum of agents and hierarchically ordered positions, and solve important welfare maximization problems subject to incentive constraints. Our results are surprising in that they provide arguments in favor of mechanisms that are well-known and debated mechanisms in the matching theory literature. For the first time as we are aware, well-studied mechanisms such as pseudomarket and Boston mechanisms are shown to be “second-best” welfare maximizers among all possible assignment rules (in the sense of being “first-best” among the incentive compatible ones.)²⁵ It is also surprising that they are not only optimal in the utilitarian sense but also in the Rawlsian (max-min) sense. This is in sharp contrast to other mechanisms such as Deferred Acceptance, which works very poorly in this context.

We have also considered an extension of our model in which the ordinal rankings of the agents for positions may be different from each other. This extension points out that the results of the main model is not just an artifact of the same ordinal ranking assumption.²⁶ In our model we abstracted away from real world features such as preexisting priorities and precedence. We plan to consider these issues in future work. We hope that the results of the present paper would be a first step for more general and interesting results that make use of optimal mechanism design tools in economic environments without money.

²⁵Which one(s) is (are) optimal depends on the parameters of the model, μ_1, μ_2 , and F . Although the conditions for optimality of these mechanisms are quite permissive, there are examples in which neither one is optimal in Utilitarian or Rawlsian problems.

²⁶Note that in this extension, DA no longer implements a Pareto-pessimal assignment rule.

6 References

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7 Appendix

Proof of Proposition 3. First of all, from Lemma (1), we know that incentive compatibility is equivalent to

$$U(v) = p(0) + \int_0^v q(t) dt$$

or

$$q(v)v + p(v) = p(0) + \int_0^v q(t) dt$$

Secondly, by first market clearing condition,

$$\int_0^1 \left(p(0) + \int_0^v q(t) dt - q(v)v \right) f(v) dv = \mu_1$$

then by changing the order of integration, we have

$$p(0) = \mu_1 + \int_0^1 f(t) J(t) q(t) dt \tag{29}$$

where

$$J(t) = t - \frac{1 - F(t)}{f(t)}$$

is the virtual valuation.

Hence, function p can be written solely as a function of q

$$p(v) = \mu_1 + \int_0^1 f(t) J(t) q(t) dt + \int_0^v q(t) dt - q(v)v$$

Note that, we have incorporated the envelope condition and market clearing condition for the top positions while writing this equality.²⁷

Therefore, a rule is incentive compatible and feasible if and only if it satisfies, (i) equation (3), (ii) monotonicity conditions, (iii) boundary conditions, and (iv) the second market

²⁷To confirm that we have incorporated the conditions, note that we can rewrite $U(v)$ as

$$U(v) = \mu_1 + \int_0^1 f(t) J(t) q(t) dt + \int_0^v q(t) dt$$

clearing condition:

$$\int_0^1 q(t) f(t) dv = \mu_2$$

Third, we argue that $p(1) \geq 0$, $p(1) + q(1) \leq 1$, and $q(0) \geq 0$ implies the rest of the boundary conditions: (i) $p(1) + q(1) \leq 1$ and $p(1) \geq 0$, then $q(1) \leq 1$, (ii) $p(1) \geq 0$ with monotonicity implies $p(0) \geq 0$; and $p(0) \geq 0$ and $q(0) \geq 0$ implies $q(0) + p(0) \geq 0$; and (iii) $p(1) + q(1) \leq 1$ and monotonicity implies $p(0) + q(0) \leq 1$, and since $q(0) \geq 0$, this implies $p(0) \leq 1$.

We have

$$\begin{aligned} p(1) &= \mu_1 + \int_0^1 f(t) J(t) q(t) dt + \int_0^1 q(t) dt - q(1) \\ &= \mu_1 + \int_0^1 f(t) \left(t - \frac{1 - F(t)}{f(t)} \right) q(t) dt + \int_0^1 q(t) dt - q(1) \\ &= \mu_1 + \int_0^1 f(t) t q(t) dt - \int_0^1 (1 - F(t)) q(t) dt + \int_0^1 q(t) dt - q(1) \\ &= \mu_1 + \int_0^1 f(t) H(t) q(t) dt - q(1) \end{aligned}$$

where

$$H(t) = t + \frac{F(t)}{f(t)}$$

is the seller's virtual valuation or information rent.

Hence, $p(1) \geq 0$ is equivalent to

$$\int_0^1 H(t) q(t) f(t) dt \geq q(1) - \mu_1$$

and utilitarian social welfare is

$$\begin{aligned} USW &= \mu_1 + \int_0^1 f(t) J(t) q(t) dt + \int_0^1 \left(\int_0^v q(t) dt \right) f(v) dv \\ &= \mu_1 + \int_0^1 f(t) J(t) q(t) dt + \int_0^1 (1 - F(t)) q(t) dv \\ &= \mu_1 + \int_0^1 f(t) \left(t - \frac{1 - F(t)}{f(t)} \right) q(t) dt + \int_0^1 (1 - F(t)) q(t) dv \\ &= \mu_1 + \int_0^1 f(t) q(t) t dt \end{aligned}$$

which confirms (1).

On the other hand, $p(1) + q(1) \leq 1$ is equivalent to

$$\int_0^1 H(t) q(t) f(t) dt \leq 1 - \mu_1$$

Lastly, we argue that if q is increasing, other monotonicity constraints (p is decreasing, $q + p$ is increasing) are automatically satisfied. Given $v > \tilde{v}$, incentive compatibility implies $p(\tilde{v}) + q(\tilde{v})\tilde{v} \geq p(v) + q(v)\tilde{v}$. Since $q(v) \geq q(\tilde{v})$ and $\tilde{v} \geq 0$, it must be the case that $p(\tilde{v}) \geq p(v)$. Also, incentive compatibility implies $p(\tilde{v}) + q(\tilde{v})v \leq p(v) + q(v)v$, or $(q(v) - q(\tilde{v}))v \geq p(\tilde{v}) - p(v)$. Since q is increasing, p is decreasing and $0 \leq v \leq 1$, we must have $q(v) - q(\tilde{v}) \geq p(\tilde{v}) - p(v)$. This implies that $p + q$ is increasing.

To sum up, p is uniquely determined given q , and the following conditions on q are necessary and sufficient conditions feasibility and incentive compatibility.

$$\begin{aligned} \int_0^1 q(t) f(t) dv &= \mu_2 \\ \int_0^1 H(t) q(t) f(t) dt &\leq 1 - \mu_1 \\ \int_0^1 H(t) q(t) f(t) dt &\geq q(1) - \mu_1 \\ q(0) &\geq 0 \\ q &\text{ is increasing} \end{aligned}$$

■

Proof of Lemma 7. We first argue that optimal q has to be a two-step function. Suppose not, that is there exists an interval $[a, b]$ such that for all $t \in (a, b)$, we have $q(a) < q(t) < q(b)$. Then consider “ironed out” version of q in the interval $[a, b]$:

$$\hat{q}(t) = \begin{cases} q(a) & \text{if } t \in [a, c] \\ q(b) & \text{if } t \in (c, b] \\ q(t) & \text{otherwise} \end{cases}$$

for c that satisfies

$$q(a)(F(c) - F(a)) + q(b)(F(b) - F(c)) = \int_a^b q(t) f(t) dt$$

It is then immediate that $\hat{q}$ satisfies MC. Also, one can check that $\int_0^1 t\hat{q}(t) f(t) dt > \int_0^1 tq(t) f(t) dt$, and $\int_0^1 J(t)\hat{q}(t) f(t) dt > \int_0^1 J(t)q(t) f(t) dt$. Moreover, with the assump-

tion that H is an increasing function, we have $\int_0^1 H(t) \hat{q}(t) f(t) dt > \int_0^1 H(t) q(t) f(t) dt$. Therefore, $\hat{q}$ also satisfies LB. Hence, we conclude that q cannot be optimal.

Then we argue that we only need to consider the following form of q

$$\tilde{q}(t) = \begin{cases} q(1) & t \in [d, 1] \\ 0 & t \in [0, d) \end{cases}$$

This is because, if there is a feasible two-step function that of the form

$$q(t) = \begin{cases} h & t \in [d, 1] \\ l & t \in [0, d) \end{cases}$$

for $h > l > 0$, (note that $1 > h$ since UB does not bind) we can find another two-step function

$$q(t) = \begin{cases} h' & t \in [d, 1] \\ l' & t \in [0, d) \end{cases}$$

with $1 > h > h'$, and $l > l'$ that also satisfies the constraints and give a higher objective function value for both problems. ■

Proof of Lemma 9. First, we argue that if UB does not bind, then q cannot be optimal unless it is a two-step function. Suppose q is not a two-step function, that is there exists an interval $[a, b]$ such that for all $t \in (a, b)$, we have $q(a) < q(t) < q(b)$. Then consider the “ironed out” version of q in the interval $[a, b]$:

$$\hat{q}(t) = \begin{cases} q(a) & \text{if } t \in [a, c] \\ q(b) & \text{if } t \in [d, b] \\ q(t) & \text{otherwise} \end{cases}$$

for c and $d \in (a, b)$ that satisfies

$$q(a)(F(c) - F(a)) + q(b)(F(b) - F(d)) = \int_a^c q(t) f(t) dt + \int_d^b q(t) f(t) dt$$

It is then immediate that $\hat{q}$ satisfies MC. Also, one can check that $\int_0^1 t \hat{q}(t) f(t) dt > \int_0^1 t q(t) f(t) dt$, and $\int_0^1 J(t) \hat{q}(t) f(t) dt > \int_0^1 J(t) q(t) f(t) dt$. Moreover, with the assumption that H is an increasing function, we have $\int_0^1 H(t) \hat{q}(t) f(t) dt > \int_0^1 H(t) q(t) f(t) dt$. Since $\hat{q}(1) = q(1)$, we can state that $\hat{q}$ also satisfies LB. And for c and d close enough to a and b , respectively, it continues to satisfy UB. Hence, we conclude that q cannot be optimal.

Next, we argue that the maximizers among the two-step functions also should satisfy UB

as equality. Consider a two-step function of the form

$$q(t) = \begin{cases} k & t \in [c, 1] \\ l & t \in [0, c) \end{cases}$$

for $c \in [0, 1]$, $0 \leq l \leq k \leq 1$.

Suppose UB does not bind, then $q(1) = k < 1$. Consider $l = 0$. Then the following assignment rule:

$$q'(t) = \begin{cases} k + \varepsilon & t \in [c + \varepsilon', 1] \\ l & t \in [0, c + \varepsilon') \end{cases}$$

by choosing $\varepsilon, \varepsilon'$ such that MC still holds, and ε , and ε' are small enough such that UB is still satisfied, clearly increases the objective function's value.

Moreover, we do not have to worry about LB in any two-step rule for a mild market when $l = 0$. MC implies $k = \frac{\mu_2}{1-F(c)}$, and LB is written as $k(1 - F(c)c) \geq k - \mu_1$. Joining both expressions we rewrite LB as $\frac{\mu_2 F(c)c}{1-F(c)} \leq \mu_1$. Since $k \leq 1$ (or $c \leq F^{-1}(1 - \mu_2)$) we have that the RHS of the previous inequality is not higher than $(1 - \mu_2)F^{-1}(1 - \mu_2)$, which is lower than μ_1 in a mild market.

Now consider $0 < l < k$. Then the following assignment rule:

$$q'(t) = \begin{cases} k & t \in [c - \varepsilon', 1] \\ l - \varepsilon & t \in [0, c - \varepsilon') \end{cases}$$

by choosing $\varepsilon, \varepsilon'$ such that MC still holds, and ε , and ε' are small enough such that UB is still satisfied, also increases the objective function's value. Since $q'(1) = q(1)$ and $\int_0^1 H(t) q'(t) f(t) dt > \int_0^1 H(t) q(t) f(t) dt$, q' also satisfies LB.

The final case involves $l = k = \mu_2$. It is easy to check that this uniform rule does not bind either UB or LB. The following rule

$$q'(t) = \begin{cases} \mu_2 + \varepsilon & t \in [F^{-1}(1/2), 1] \\ \mu_2 - \varepsilon & t \in [0, F^{-1}(1/2)) \end{cases}$$

for $\varepsilon > 0$ small enough, meets MC, UB and LB. q' clearly improves over the uniform rule $q(t) = \mu_2$.

Hence, we have shown that no rule can be optimal if UB does not bind. ■

Proof of Lemma 10. We have $H'(c) = 2 - \frac{f'(c)F(c)}{f(c)^2}$ and $T'(c) = \frac{f(c)}{F(c)}[c - T(c)] = \frac{f(c) \int_0^c F(x) dx}{F(c)^2}$. Now, $T''(c)/T'(c) = \frac{f'(c)}{f(c)} + \frac{F(c)}{\int_0^c F(x) dx} - 2\frac{f(c)}{F(c)}$. Multiplying by $\frac{F(c)}{f(c)}$ we obtain that the sign of $T''(c)$ is the sign of $Z(c) = 1/T'(c) - H'(c)$.

Notice that $\lim_{c \rightarrow 0} Z(c) = 0$ (after applying l'Hôpital's rule), thus around $c = 0$, $T'(c)$ does not vary while $H'(c)$ increases by strict convexity of H . Then, for some $\varepsilon > 0$ and every $c \in (0, \varepsilon)$ we have $T''(c) < 0$ (since $Z(c) < 0$). Since Z is continuous on $(0, 1)$, we cannot have $c' \in (0, 1)$ such that $Z(c') > 0$. If such c' exists then by continuity of Z there is $c'' \in (\varepsilon, c')$ such that $Z(c'') = 0 < Z'(c'')$. But again $Z(c'') = 0$ implies $T''(c'') = 0$ and given $H'' > 0$ we get $Z'(c'') < 0$, a contradiction. Hence, Z is negative on the domain $(0, 1)$, thus T is strictly concave (and S is decreasing). ■

Proof of Lemma 11. First, note that no feasible R can first-order stochastically dominate, or dominated by, either R^* or R^{**} . This is because, a domination implies (since H is increasing,) $E_R H(v) \neq \frac{1-\mu_1}{\mu_2}$.

This implies $q(0) < l^{**}$ for all feasible rules other than q^{**} (otherwise R would be first-order stochastically dominated by R^{**}). Moreover, for all feasible rules other than q^* , we have to have $q(1) > k^*$ (otherwise R would be first-order stochastically dominated by R^*).

Another implication is: If $q(v) = 0$ for almost all $v \leq d^*$ and $q \neq q^*$, q is not feasible; if $q(v) = 1$ for almost all $v \geq \frac{\mu_1}{1-\mu_2}$ and $q \neq q^{**}$, q is not feasible.

From the previous two lines, we can establish that if q is feasible and is not either q^* or q^{**} , both functions $[r(v) - r^*(v)]$ and $[r(v) - r^{**}(v)]$ switch sign exactly twice.

Then, if $E_R v = E_{R^*} v$, we have that R either second-order dominates, or dominated by, R^* . The same happens if $E_R v = E_{R^{**}} v$. ■

Proof of lemma 13. We show point 1), since point 2) follows from analogous arguments. We have $H'(c) = 2 - \frac{f'(c)F(c)}{f(c)^2}$ and $\tilde{T}'(c) = 1 - \frac{F(c)-f(c)c}{F(c)^2}$. Now, $\tilde{T}''(c) = \frac{f'(c)c}{F(c)^2} + \frac{2f(c)[F(c)-f(c)c]}{F(c)^3}$. Multiplying by $\frac{F(c)^3}{f(c)^2c}$ we obtain that the sign of $\tilde{T}''(c)$ is the sign of $\tilde{Z}(c) = 2\frac{F(c)}{f(c)c} - H'(c)$. Define $N(c) = \frac{F(c)}{f(c)c}$ and notice that $N'(c) = \frac{1}{c} [H'(c) - N(c) - 1] = \frac{1}{c} [N(c) - 1 - \tilde{Z}(c)]$. So if $\tilde{Z}(c) = 0$, then $N'(c) < 0$ since $c/F(c)$ decreasing implies $N(c) < 1$.

Notice that $\lim_{c \rightarrow 0} \tilde{Z}(c) = 0$ (after applying l'Hôpital's rule). Thus around $c = 0$, $N(c)$ decreases while $H'(c)$ increases by strict convexity of H . Then, for some $\varepsilon > 0$ and every $c \in (0, \varepsilon)$ we have $\tilde{T}''(c) < 0$ (since $\tilde{Z}(c) < 0$). Since $\tilde{Z}$ is continuous on $(0, 1)$, we cannot have $c' \in (0, 1)$ such that $\tilde{Z}(c') > 0$. If such c' exists then by continuity of $\tilde{Z}$ there is $c'' \in (\varepsilon, c')$ such that $\tilde{Z}(c'') = 0 < \tilde{Z}'(c'')$. But again $\tilde{Z}(c'') = 0$ implies again that $N(c)$ decreases, and given $H'' > 0$ we get $\tilde{Z}'(c'') < 0$, a contradiction. Hence, $\tilde{Z}$ is negative on the domain $(0, 1)$, thus $\tilde{T}$ is strictly concave (and $\tilde{S}$ is decreasing). ■

Proof of Proposition 14. First, in the simple maximization problem (16) above, the “benefit to cost ratio” of high positions is $\frac{1}{\pi_1}$, whereas it is $\frac{v}{\pi_2}$ for the medium positions.

Therefore, as long as the constraint $p + q \leq 1$ is not binding, agents with $v < v^*(\pi_1, \pi_2) \equiv \frac{\pi_2}{\pi_1}$ would want to buy from high positions, and agents with $v > v^*(\pi_1, \pi_2) \equiv \frac{\pi_2}{\pi_1}$ would want to buy from medium positions. We already argued that $\pi_1 \geq 1$, if we also have $\pi_2 \geq 1$, then agents would not have any money left for buying from their inferior positions (and $p + q \leq 1$ does not bind). We present this case as Case 1 below, which turns out to be corresponding to the tough market. On the other hand, if we have $\pi_2 < 1$, then agents with $v > \frac{\pi_2}{\pi_1}$ would have money left after buying the whole probability of medium positions ($q(v) = 1$). We present this case as Case 2 below, which turns out to be corresponding to the mild market.

Case 1. Middle-class positions are expensive: $\pi_2^* \geq 1$.

Suppose $\pi_2^* \geq 1$ in a CE, and let unique indifferent type be v^* (which will be $v^*(\pi_1, \pi_2)$ equal to $\frac{\pi_2}{\pi_1}$.) We can observe that types below v^* only buy high positions ($p = \frac{1}{\pi_1}$, $q = 0$), and types above v^* only buys medium positions ($p = 0$, $q = \frac{1}{\pi_2}$). For this to be a CE, we have to have

$$\frac{F(v^*)}{\pi_1} = \mu_1$$

and

$$\frac{1 - F(v^*)}{\pi_2} = \mu_2$$

These two imply

$$\frac{\pi_2}{\pi_1} = \frac{\frac{1 - F(v^*)}{\mu_2}}{\frac{F(v^*)}{\mu_1}}$$

Since $\frac{\pi_2}{\pi_1} = v^*(\pi_1, \pi_2)$, we can conclude that a CE can exist only if there is a type $v^* \in (0, 1)$ such that

$$\frac{F(v^*)v^*}{1 - F(v^*)} = \frac{\mu_1}{\mu_2} \quad (30)$$

It is immediate to see that this type exists and is unique. This is because $\frac{F(v)v}{1 - F(v)}$ is strictly increasing with range $[0, \infty)$. However, it does not mean that this type of a CE necessarily exists. One must check the condition $\pi_2^* \geq 1$, or $\mu_2 \leq 1 - F(v^*)$, or $v^* \leq F^{-1}(1 - \mu_2)$. Given that the LHS of the equation (30) is monotonically increasing, this CE exists if and only if

$$\frac{(1 - \mu_2)F^{-1}(1 - \mu_2)}{\mu_2} \geq \frac{\mu_1}{\mu_2}$$

or $\mu_1 - (1 - \mu_2)F^{-1}(1 - \mu_2) \leq 0$, which is exactly when the economy is “tough.”

Hence, in a tough market, the CE prices are

$$\pi_1^* = \frac{F(v^*)}{\mu_1}$$

$$\pi_2^* = \frac{1 - F(v^*)}{\mu_2}$$

where v^* uniquely solves (30).

In this CE, the assignment of probabilities is as follows. Types $v < v^*$ obtain the probability bundle

$$\left(\frac{\mu_1}{F(v^*)}, 0, 1 - \frac{\mu_1}{F(v^*)}\right),$$

types $v > v^*$ get

$$\left(0, \frac{\mu_2}{1 - F(v^*)}, 1 - \frac{\mu_2}{1 - F(v^*)}\right).$$

where the first, second, and third components represents probability shares of high, medium and low positions.

Case 2. Middle-class positions are cheap: $\pi_2^* < 1$.

Suppose $\pi_2^* \geq 1$ in a CE, and let unique indifferent type be v^* . Types below v^* only buy high positions ($p = \frac{1}{\pi_1}$, $q = 0$). On the other hand, types above v^* buy medium positions, and buy high positions with the rest of their money in such a way that the total probability of being assigned to high or medium positions do not exceed 1 ($p + q \leq 1$). The solution to high types' maximization problem can be shown to be given by $p = \frac{1 - \pi_2}{\pi_1 - \pi_2}$ and $q = \frac{\pi_1 - 1}{\pi_1 - \pi_2}$.²⁸ Also, as in Case 1, we can argue that the unique indifferent type $v^*(\pi_1, \pi_2)$ is again equal to $\frac{\pi_2}{\pi_1}$.²⁹ For this to be a CE, we have to have

$$\frac{F(v^*)}{\pi_1} + (1 - F(v^*)) \frac{1 - \pi_2}{\pi_1 - \pi_2} = \mu_1$$

$$(1 - F(v^*)) \frac{\pi_1 - 1}{\pi_1 - \pi_2} = \mu_2$$

Above system of two linear equations have the following unique solution in prices, as a

²⁸The maximization problem is

$$\max p + qv$$

subject to

$$\pi_1 p + \pi_2 q \leq 1 \text{ and } p + q \leq 1$$

we can argue that at the optimal solution both constraints have to bind, and $p = \frac{1 - \pi_2}{\pi_1 - \pi_2}$ and $q = \frac{\pi_1 - 1}{\pi_1 - \pi_2}$ solve them simultaneously and therefore is the maximizer.

²⁹If we compare the utility at the optimal bundle of lower types and higher types. We observe that again the critical type is $v^*(\pi_1, \pi_2) = \frac{\pi_2}{\pi_1}$ as $\frac{1 - \pi_2}{\pi_1 - \pi_2} + v \frac{\pi_1 - 1}{\pi_1 - \pi_2} > \frac{1}{\pi_1}$ if and only if $v > \frac{\pi_2}{\pi_1}$.

function of v^* , μ_1 and μ_2 .

$$\begin{aligned}\pi_1^* &= \frac{F(v^*)}{F(v^*) - (1 - \mu_1 - \mu_2)} \\ \pi_2^* &= \frac{1}{\mu_2} \frac{(1 - \mu_1)F(v^*) - (1 - \mu_1 - \mu_2)}{F(v^*) - (1 - \mu_1 - \mu_2)}\end{aligned}$$

Since $v^* = \frac{\pi_2^*}{\pi_1^*}$, we have to have

$$v^* = \frac{\pi_2^*}{\pi_1^*} = \frac{1 - \mu_1}{\mu_2} - \frac{1 - \mu_1 - \mu_2}{\mu_2 F(v^*)}$$

or

$$(1 - \mu_1 - v^* \mu_2) F(v^*) = 1 - \mu_1 - \mu_2 \quad (31)$$

The question is, whether there exists a v^* that solves (31), and whether for that v^* we would have $\pi_2^* < 1$.

Consider the function $G(v) \equiv (1 - \mu_1 - v\mu_2)F(v) - (1 - \mu_1 - \mu_2)$. Observe that, (i) $G(0) < 0$, and (ii) $G(F^{-1}(1 - \mu_2)) = (1 - \mu_1 - \mu_2 F^{-1}(1 - \mu_2)) (1 - \mu_2) - (1 - \mu_1 - \mu_2) > 0$. The latter is due to $F^{-1}(1 - \mu_2) < \frac{\mu_1}{1 - \mu_2}$ and the fact that $(1 - \mu_1 - \frac{\mu_1 \mu_2}{1 - \mu_2}) (1 - \mu_2) = 1 - \mu_1 - \mu_2$. Since G is continuous, the intermediate value theorem applies: there exists a v with $G(v) = 0$.

We can also show uniqueness of v^* . We have $G'(v) = f(v)(1 - \mu_1 - v\mu_2) - \mu_2 F(v)$. Hence, its sign is equivalent to that of $G'(v)/f(v) = 1 - \mu_1 - \mu_2 H(v)$ which is a decreasing function (since $H(\cdot)$ is increasing). We therefore conclude that G is concave. Since G is concave, $G(0) < 0$, $G(F^{-1}(1 - \mu_2)) > 0$, and $G(1) = 0$, we can conclude that there exists a unique v^* that satisfies $G(v^*) = 0$ (and also $v^* \in (0, F^{-1}(1 - \mu_2))$).

Lastly, we show that under a mild market, we would also have $\pi_2^* < 1$. This follows because $\frac{1}{\mu_2} \frac{(1 - \mu_1)F(v^*) - (1 - \mu_1 - \mu_2)}{F(v^*) - (1 - \mu_1 - \mu_2)} < 1$ reduces to $1 - F(v^*) > \mu_2$, or $v^* < F^{-1}(1 - \mu_2)$, which we already argued. On the other hand, under a tough market we would have $G(F^{-1}(1 - \mu_2)) > 0$, which implies $v^* \geq F^{-1}(1 - \mu_2)$, and $\pi_2^* \geq 1$.

In conclusion, under a mild market, we would have $\pi_2^* < 1$ and in CE, the assignment of probabilities are as follows. Types $v < v^*$ obtain the probability bundle

$$(1 - \frac{1 - \mu_1 - \mu_2}{F(v^*)}, 0, \frac{1 - \mu_1 - \mu_2}{F(v^*)}),$$

types $v > v^*$ get

$$(1 - \frac{\mu_2}{1 - F(v^*)}, \frac{\mu_2}{1 - F(v^*)}, 0)$$

where the first, second, and third components represents probability shares of high, medium and low positions. ■

Proof of Proposition 15. Obviously we must only consider cases when $m_1 \geq \mu_1$ (if $m_1 < \mu_1$ using strategy 2 would never be a best response), in which we say that high-class positions are *overdemanded*. So we are left with two cases to analyze: one in which middle-class positions are *overdemanded* ($m_2 \geq \mu_2$), and the case in which they are *underdemanded* ($m_2 < \mu_2$). It turns out that the former case corresponds to the tough market and latter to mild market.

Case 1. Middle-class positions are overdemanded: $m_2 \geq \mu_2$.

In this case it is clear that those who are rejected in the first round end up being assigned to low-class positions. Those choosing strategy $j \in \{1, 2\}$ obtain μ_j/m_j chances at the position ranked first and $1 - \mu_j/m_j$ chances for a low-class position. An indifferent type $v^*(m_1, m_2)$ meets $\mu_1/m_1 = v^*(m_1, m_2)\mu_2/m_2$. Types above the indifferent type best-respond by choosing strategy 2, and types below rather choose strategy 1. Therefore a Nash Equilibrium (NE) is characterized by the existence of a $v^* \in (0, 1)$ such that

$$\frac{F(v^*)v^*}{1 - F(v^*)} = \frac{\mu_1}{\mu_2}$$

As in Case 1 of the proof of Proposition 14, a solution to this equation exists and is unique. It remains to check that in effect $m_2 > \mu_2$, or $\mu_2 < 1 - F(v^*)$, or $v^* < F^{-1}(1 - \mu_2)$. Given that the LHS of the equation above is monotonically increasing, a Case 1 NE exists if and only if

$$\frac{(1 - \mu_2)F^{-1}(1 - \mu_2)}{\mu_2} > \frac{\mu_1}{\mu_2}$$

or $\mu_1 - (1 - \mu_2)F^{-1}(1 - \mu_2) \leq 0$, which exactly is when the market is “tough.”

In this bayesian Nash equilibrium, Types $v < v^*$ obtain the probability bundle

$$(\frac{\mu_1}{F(v^*)}, 0, 1 - \frac{\mu_1}{F(v^*)}),$$

whereas types $v > v^*$ get

$$(0, \frac{\mu_2}{1 - F(v^*)}, 1 - \frac{\mu_2}{1 - F(v^*)}).$$

Note that in Cases 1, both the pseudomarket (CE) and the Boston mechanism (NE) obtain the same random assignment. This is not a coincidence. Miralles (2008) shows that both mechanisms obtain the same assignment in cases where only one kind of positions is underdemanded.

Case 2. Middle-class positions are underdemanded: $m_2 < \mu_2$.

In this case those agents who play strategy 2 are assigned to middle-class positions with certainty. Agents playing strategy 1 obtain $\frac{\mu_1}{m_1}$ chances at the high-class positions in the first assignment round, $\frac{\mu_2 - m_2}{m_1}$ chances (in the second round) for middle-class positions, and $\frac{1 - \mu_1 - \mu_2}{m_1}$ for the low-class positions in the last round.

Let v^* be the type who is indifferent between the two strategies. We would have

$$v^* = \frac{\mu_1}{m_1} + \frac{\mu_2}{m_1} v^*$$

hence,

$$v^* = \frac{\mu_1}{m_1 - \mu_2 + m_2} = \frac{\mu_1}{1 - \mu_2}$$

(since $m_1 + m_2 = 1$).

Again, in equilibrium, types $v > v^*(m_1, m_2) = \frac{\mu_1}{1 - \mu_2}$ choose strategy 2, and types $v < \frac{\mu_1}{1 - \mu_2}$ choose strategy 1. Hence, $m_1 = F(\frac{\mu_1}{1 - \mu_2})$ and $m_2 = 1 - F(\frac{\mu_1}{1 - \mu_2})$.

We now check that, $m_2 < \mu_2$ holds. That is, $1 - F(\frac{\mu_1}{1 - \mu_2}) < \mu_2$. This is equivalent to the market being “mild” $\mu_1 - (1 - \mu_2)F^{-1}(1 - \mu_2) > 0$.

Finally, the obtained probability bundles are

$$(\frac{\mu_1}{F(\frac{\mu_1}{1 - \mu_2})}, 1 - \frac{1 - \mu_2}{F(\frac{\mu_1}{1 - \mu_2})}, \frac{1 - \mu_1 - \mu_2}{F(\frac{\mu_1}{1 - \mu_2})})$$

for types $v > \frac{\mu_1}{1 - \mu_2}$, and

$$(0, 1, 0)$$

for types $v < \frac{\mu_1}{1 - \mu_2}$. ■