Sequential versus Simultaneous Assignment Systems and Two Applications*

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Abstract

We study matching markets in practice, where a set of objects are to be assigned to a set of agents sequentially in two rounds. The placement of students in exam and mainstream schools in the U.S. and the appointment of teachers to the state schools in Turkey until recently are two examples of such markets. We analyze the mechanisms currently in use in both markets and show that they fail to satisfy desirable fairness and welfare criteria. Moreover, they give participants perverse incentives: misreporting preferences can be beneficial and improved performance on the admissions test may worsen a participant's assignment. We show that these shortcomings are inherent in more general sequential assignment systems as well, which motivates us to propose an alternative simultaneous assignment system, applicable to both markets, through which assignments take place in a single round. Our analysis may also shed light on the recent reform in the Turkish teacher appointment system.

1 Introduction

Simultaneous allocation systems, whereby distribution of all resources takes place in a single round, are widely used for solving static allocation problems. On the other hand, when faced with subtle allocational constraints, policy makers often resort to sequential allocation systems whereby different sets of resources

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are distributed in different rounds. In this paper, we study sequential allocation systems in the context of indivisible good assignment and assess the advantages and disadvantages of such a system when contrasted with an analogous simultaneous allocation system.

In a sequential assignment problem, a set of agents is to be assigned a set of objects in a sequential fashion in (at least) two rounds, and each agent is entitled to receive at most one object. More specifically, there is a first round of assignments in which all agents actively participate by reporting their rank-ordered preferences when only a subset of objects is available. The first round is then followed by a second round of assignments in which all remaining objects are assigned to those agents who were unassigned in the first round. We are motivated by two applications of this problem in practice: student placement to exam and regular public schools in the U.S. and the appointment of teachers to state schools in Turkey.

In Boston and New York City, there are two types of public schools: exam and regular (mainstream) schools.[1] Students who wish to apply to exam schools take a centralized test and are then ranked based on their scores. Meanwhile, regular schools rank students based on certain predetermined criteria, e.g., proximity and sibling status. The admissions for both type of schools are processed separately. In general, the admissions decisions for the exam schools are completed well before any students are assigned to the regular schools. In particular, students are assigned to exam schools via a serial dictatorship mechanism, and the unassigned students are then assigned to regular schools via a student-proposing deferred acceptance (DA) mechanism (Gale and Shapley, 1962) in a second round. (See the Appendix for a detailed description of the assignment systems in Boston and New York City.)

In Turkey, the assignment of teachers to teaching positions in state schools takes place via a centralized process overseen by the Turkish Ministry of Education (TMoE). Every year the TMoE offers a standardized test to those university graduates who wish to serve in state-sponsored jobs. Although this test is taken mostly by new university graduates, many who have graduated in the past are also eligible to take it if they wish to do so.[2] In a given year, the appointment of teachers to state schools is based solely on the candidates' performance on that year's test. There are two types of teaching positions in each specialization: *tenured* positions, which offer a life-time employment guarantee, and *contractual* positions, which can be held only for a fixed number of years (typically for only a few years). The conditions of employment are based on a specific contract mutually agreed upon by the school and the teacher. Although an otherwise identical

[1] There are different types of regular schools.
[2] Some of them may be currently employed as a teacher.

tenured position is generally preferable to a contractual position, it is also common to observe strong preference for contractual positions in major metropolitan cities such as Istanbul over tenured positions in smaller cities or rural areas.

In a given year, the TMoE first announces the list of all available tenured positions in each school and each specialization throughout Turkey. Then each applicant, be it a new graduate or an existing contractual teacher, submits rank-ordered preference lists over the available tenured positions before a certain deadline announced by the ministry.[3] In this first round, existing contractual teachers who are seeking a new position are also restricted to rank-list only tenured positions.[4] Applicants are then assigned to the available positions by a serial dictatorship mechanism induced by the test scores. If a contractual teacher is unassigned in the first round, then she retains her current job assignment. Otherwise, she is appointed to a tenured position and a contractual position at her old school opens up for a new appointment. Typically within a few weeks after the first round, the TMoE announces the list of available contractual positions. In this second round, only the unassigned new graduates are permitted to apply to these contractual positions. Applicants are again assigned via a serial dictatorship mechanism induced by the test scores. (See Table 1 in the Appendix for some summary statistics of this system in recent years.)

We show that the multiple-round student assignment system in the U.S. and the teacher appointment system in Turkey share a number of serious deficiencies. Among other shortcomings, both systems fail to generate Pareto-efficient or fair assignments, and both systems induce strategic action on the part of applicants while deciding what preferences to submit in each round. We then ask whether such shortcomings can be overcome by alternative systems and turn to investigate general sequential assignment systems, in which the mechanisms used in each round satisfy certain properties that are by and large deemed desirable in the matching literature.

We argue that the deficiencies of the systems in the U.S. and Turkey are not specific to these particular contexts. Our analysis indicates that there may indeed be a fundamental problem with achieving distributional and strategic goals via sequential assignment systems that employ mechanisms that satisfy even very mild requirements. Our results are as follows:

If $\Psi = (\varphi^1, \varphi^2)$ is a straightforward[5] sequential assignment system, then it is also wasteful (Theorem

[3] Existing teachers employed in tenured positions are not allowed to participate in this assignment procedure.

[4] In other words, any contractual position currently filled by an applicant cannot be rank-listed by any applicants including its current occupant.

[5] A system is *straightforward* if it cannot be manipulated by an agent via a pair of misreports, one for each round. We formulate this requirement as a natural extension of the strategy-proofness property to a sequential system.

??). That is, any straightforward system necessarily generates inefficient outcomes. If $\Psi = (\varphi^1, \varphi^2)$ is a straightforward sequential assignment system that employs non-wasteful mechanisms in both rounds, then such a system cannot generate fair outcomes (Theorem **??**). Similarly, if $\Psi = (\varphi^1, \varphi^2)$ is a straightforward sequential assignment system that employs non-wasteful mechanisms in both rounds and φ^1 selects a fair outcome whenever all objects prioritize agents in the same order, then Ψ does not respect improvements in the priority order (Theorem **??**). If $\Psi = (\varphi^1, \varphi^2)$ is a sequential assignment system that employs mechanisms φ^1 and φ^2 in rounds 1 and 2 such that φ^1 is individually rational and non-wasteful, and φ^2 is non-wasteful, then such a system is prone to manipulation (Theorem **??**).

We also characterize the subgame perfect Nash equilibria (SPNE) induced by a sequential preference revelation game of a sequential assignment system. We find that when both φ^1 and φ^2 are individually rational, non-wasteful, and either [population monotonic and non-bossy] or fair, then every SPNE outcome of the preference revelation game associated with system Ψ leads to a non-wasteful and individually rational matching (Theorem **??**). On the other hand, when both φ^1 and φ^2 are individually rational, non-wasteful, population monotonic, and minimally fair, then every SPNE outcome of the preference revelation game associated with system Ψ leads to a matching that does not induce any priority violations (Theorem **??** and **??**). As corollaries of these results, we provide a detailed account of the characteristics of the set of SPNE for each of the two applications that motivate our study.

Our analysis points to clear disadvantages of sequential assignment systems and provides justification for the alternative use of single-round assignment systems when possible. This conclusion is also supported by the recent transition of the TMoE from the system analyzed here to a simpler single-round simultaneous assignment system.[6] More broadly, these observations motivate us to advocate the use of a suitable adaptation of Gale and Shapley's celebrated *deferred acceptance* (DA) mechanism to the specific context as a single round assignment system. In particular, in the context of teacher assignment, the dominant strategy outcome of DA Pareto dominates any SPNE of the old assignment system.

1.1 Related Literature

The main characteristic that distinguishes the type of problems we study here from the vast majority of the problems considered in the literature is that they involve sequential assignment of indivisible resources. Whereas the set of agents and resources are predetermined and fixed in a standard, single-round simul-

[6]As far as we are aware, this transition took place without the involvement of any economists in the decision process.

taneous assignment problem, in a sequential assignment problem agents and resources considered within a round may depend on decisions made in a previous round. Differently put, in a sequential assignment problem, an agent might have the ability to choose which round he is to get which assignment. Yet, the two types of problem still share similar strategic and distributional objectives.

There is now an extant literature on school choice plans, but as far as we are aware, virtually all of these models abstract away from the multi-round nature of these problems and focus exclusively on a single-round simultaneous assignment system. Still, we are not the first to point out the deficiencies of a sequential student assignment system. In their detailed examination of NYC student assignment plan, Abdulkadiroğlu, Pathak, and Roth (2009) argue that the current multi-round assignment plan may result in unstable student assignments. They write:

We would have preferred to integrate these two rounds into one, by having applicants include the specialized schools in their preference lists. (The two-round design creates a possibility of unstable allocations, as when a student gets an offer from a specialized school, but not from a nonspecialized school he prefers that would have had a place for him after the specialized-school students have declined places.)

While Abdulkadiroğlu, Pathak, and Roth (2009) caution against potential stability issues resulting from the sequential system in New York City, in the present paper we identify potential incentive and welfare issues associated with such systems.

On the other hand, the *teacher assignment problem* (TAP) in Turkey, described above, has features reminiscent of the *student placement problem* (SPP) due to Balinksi and Sönmez (1999) and the *house allocation problem with existing tenants* (HAPwET) due to Abdulkadiroğlu and Sönmez (1999). As in the context of SPP, in TAP too, applicants are ranked based on their test scores, and *fairness* (i.e., favoring applicants with better test scores) is a central goal. And, as in HAPwET, some of the applicants–the contractual teachers–have private endowments–the contractual positions they currently occupy–that may later become available for reassignment to other applicants.

Another paper that is related to ours is Ergin and Sönmez (2006), where the authors characterize the set of NE of the widely used Boston mechanism and show that this set coincides with the set of stable matchings. We find that while this conclusion need not hold generally for any sequential assignment problem, in the context of TAP (but not in that of SCPwERS) the set of SPNE of the sequential preference revelation game is also equal to the stable set.[7]

The only paper (that we are aware of) to consider sequential assignment is Westkamp (2012), where the author studies the German college admissions system which operates through a combination of the Boston and the college-proposing deferred acceptance mechanism.[8] Similarly to Ergin and Sönmez (2006), Westkamp also characterizes the set of SPNE of this game as being the stable set. While we also provide characterizations of SPNE for both of our applications, we show that the equivalence of SPNE to the stable set in general may not always be guaranteed. Most notably, in contrast to Westkamp, our focus here is on properties of a general sequential assignment system and on showing that the sources of the deficiencies may be inherently related to the sequential nature of the assignment system. As such, we show that these deficiencies may be impossible to avoid regardless of what specific mechanism is used in any round. Another difference is that we consider sequential systems composed of strategy-proof mechanisms, whereas Westkamp analyzes a particular sequential assignment system in which both rounds feature mechanisms that are vulnerable to manipulation. Hence, we show that the strategic vulnerabilities of sequential systems cannot be overcome by *any* system even though strategy-proofness is guaranteed within each round.[9]

Braun et al. (2011) compare the performance of the sequential German college admissions systems and a modified version of the DA mechanism through experiments. The results of the experiments show that the current practice in Germany harms the high-performing students and creates incentives for them to misreport their preferences. On the other hand, the modified DA mechanism improves the welfare of the high-performing applicants.

The rest of the paper is organized as follows. Section 2 introduces the formal model. Section 3 provides a detailed description of the sequential systems in the U.S. and Turkey. Section 4 presents impossibility results concerning general sequential systems. Section 5 characterizes the SPNE of general sequential systems as well as those of the two motivating applications. Section 6 presents a simple alternative to sequential systems. Section 7 concludes.

[7]In the context of TAP, stability is characterized by the combination of individual rationality, fairness, and nonwastefulness.

[8]A major reason behind the current two-round German college admissions system is to accommodate affirmative action considerations.

[9]Westkamp conjectures that the incentive issues observed in the current German college admissions system may not be solved by adopting another sequential system.

2 Model

Let $I^* = \{i_1, i_2, ..i_n\}$ be the set of all agents, $S^* = \{s_1, s_2, ..., s_m\}$ be the set of all objects, and $q^* = (q_{s_1}, q_{s_2}..., q_{s_m})$ be the capacity vector for all objects. Let $\emptyset$ represent the being unassigned option for both agents and objects. Let $\succ^* = (\succ_{s_1}, \succ_{s_2}, ..., \succ_{s_m})$ denote a priority profile, where $\succ_s$ is the strict priority order for object s such that $\emptyset \succ_s i$ means that agent i is not acceptable for object s. We allow for an object to be socially or privately owned. Let $h^* = (h(i))_{i \in I^*}$ be an ownership profile, where $h(i)$ is the object for which agent i has the property right (i.e., her endowment) such that $h(i) = \emptyset$ means that agent i has no property right over any object. Each agent i can own at most one object, i.e. $|h(i)| \leq 1$. On the other hand, an object can be owned by more than one agent. In particular, an object $s \in S$ can be owned by at most q_s agents. Unless stated otherwise, an object s is owned by either 0 or q_s agents.

Each agent i has a strict (i.e., complete, transitive, and antisymmetric) preference relation P_i over $S \cup \{\emptyset\}$. Let R_i denote the associated "at least asgood as" relation of agent i. We thus have

$$sR_is' \Leftrightarrow sP_is' whenever s \neq s'.$$

Let $\succ$ and $\mathbf{P}$ be the sets of all possible priority and preference profiles. A **sequential assignment problem**, or a *problem* for short, is a 6-tuple $(I, S, P, q, \succ, h)$ where $I \subseteq I^*$, $S \subseteq S^*$, $P = (P_i)_{i \in I} \in \mathbf{P}$ is a preference profile, $q = (q_s)_{s \in S}$, $\succ = (\succ_s)_{s \in S} \in \succ$, and $h = (h(i))_{i \in I}$.

Fix a problem $(I, S, P, q, \succ, h)$. A matching is a function $\mu : I \to S \cup \emptyset$ such that the number of agents assigned to an object s does not exceed the total number of the copies of s and each agent can be assigned to at most one object, i.e., $|\mu^{-1}(s)| \leq q_s$ and $|\mu(i)| \leq 1$ for all $s \in S$ and $i \in I$. Let $\mathcal{M}$ be the set of all matchings. A matching μ is **non-wasteful (NW)** if there exists no agent-object pair (i, s) such that $|\mu^{-1}(s)| < q_s$, $i \succ_s \emptyset$ and $s \ P_i \ \mu(i)$.[10] A matching μ is **individually rational (IR)** if no agent is assigned to an object that either she finds worse than being unassigned or she is unacceptable for. Formally, a matching μ is individually rational if $\mu(i) R_i \emptyset$ and $i \succ_{\mu(i)} \emptyset$ for all $i \in I$. A matching μ **Pareto dominates** another matching μ' if each agent weakly prefers her assignment in μ to her assignment in μ' and at least one agent i strictly prefers her assignment in μ to her assignment in μ'. A matching μ is **Pareto efficient**

[10] Our definition slightly differs from the standard non-wastefulness notion (see Balinski and Sonmez (1999)). Here, we also add the condition that $i \succ_s \emptyset$. In the standard student placement or school choice problem all agents are acceptable for objects. In our case, if a student is unacceptable to a school, then the unfilled seats for that school are not considered as wasted.

if it is not Pareto dominated by another matching μ'. A matching μ is **fair** if whenever an agent prefers some other agent's assignment to her own, then the other agent has a higher priority for that object than herself. Formally, if μ is fair then for every $i, j \in I$, $\mu(j) P_i \mu(i)$ implies $j \succ_{\mu(j)} i$. A matching μ is **stable** if it is non-wasteful, individually rational, and fair. A matching μ is **mutually fair (MF)** if there does not exist an agent-object pair (i, s) such that (1) i ranks s at the top of his preference list, (2) $\mu(i) \neq s$, and (3) there exists another agent i' with lower priority for s than i with $\mu(i') = s$. Let $r(P_i, s)$ be the rank of s in the preference list P_i. A matching μ **favors higher ranks (FHR)** if, if i is assigned to a worse object than s, then all agents assigned to s rank s at least as high as i. Formally, μ favors higher ranks if whenever there exists an agent-object pair (i, s) such that $sP_i\mu(i)$ then $r(P_i, s) \geq r(P_j, s)$ for all $j \in \mu^{-1}(s)$.[11]

A **mechanism** φ is a mapping that associates a matching to a given problem. Denote the outcome selected by mechanism φ for problem $(I, S, P, q, \succ, h)$ by $\varphi(I, S, P, q, \succ, h)$ and the match of agent $i \in I$ by $\varphi_i(I, S, P, q, \succ h)$.

A mechanism is **non-wasteful {(mutually) fair}** [**individually rational**] **¡favors higher ranks¿** if its outcome is non-wasteful {(mutually) fair} [individually rational] ¡able favoring higher ranks¿ in a given problem.

A mechanism φ is **weakly non-bossy (WNB)** if for any $P = (P_j)_{j \in I}$ and P_i' if $\varphi_i(I, S, P, q, \succ, h) = \varphi_i(I, S, (P_i', P_{-i}), q, \succ, h) = \emptyset$ then $\varphi(I, S, P, q, \succ, h) = \varphi(I, S, (P_i', P_{-i}), q, \succ, h)$.

A mechanism φ is **resource monotonic (RM)** if $\varphi_i(I, S, P, q, \succ, h) R_i \varphi_i(I, S, P, (q_s', q_{-s}), \succ, h)$ for all $s \in S$, $q_s' \leq q_s$, $i \in I$ and $P \in \mathbf{P}$.

A mechanism φ is (weakly) **population monotonic (PM)** if $\varphi_j(I, S, (P_i', P_{-i}), q, \succ, h) R_j \varphi_j(I, S, P, q, \succ, h)$ for all $j \in I \setminus \{i\}$ where ($\varphi_j(P) = \emptyset$ and) $\emptyset P_i' x$ for all $x \in S$.[12]

A mechanism φ is **monotonic** if it is resource and population monotonic.

A mechanism φ is **independent of irrelevant agents (IIA)** if $\varphi_i(I \diagdown \{k\}, S, (P_j)_{j \in I \diagdown \{k\}}, q, \succ h) = \varphi_i(I, S, P, q, \succ h)$ where $P_k : \emptyset P_k x$ for all $x \in S$, $i \in I \diagdown \{k\}$ and $P \in \mathbf{P}^I$.

A mechanism φ is **strategy-proof (SP)** if it is always a dominant strategy for each agent to report

[11] Favoring higher ranks was introduced by Kojima and Ünver (2013). In contrast to them, we do not require all the seats of school s to be filled.

[12] Our population monotonicity definition is similar to that of Kojima and Ünver (2013). In contrast to their definition, the assignment of agent i is not removed from the problem in our specification.

his preferences truthfully. Formally, for every $i \in I$ and every P_i', and every P, we have

$$\varphi_i(I, S, P, q, \succ, h) R_i \varphi_i(I, S, P_i', P_{-i}, q, \succ, h).$$

In Table 1, we summarize the performance of the well-known mechanisms based on the axioms defined.[13]

	NW	IR	Fair
Object-proposing DA	✓	✓	✓
Agent-proposing DA	✓	✓	✓
Top Trading Cycles	✓	✓	
Boston Mechanism	✓	✓	
Serial Dictatorship	✓	✓	✓The serial dictatorship mechanism is defined in an environment where all obje

The serial dictatorship mechanism is defined in an environment where all objects have the same priority order. In this environment the outcome of the serial dictatorship is fair.

Table 1. Performance of Mechanisms

A **sequential assignment system,** or a *system* for short, is a pair of mechanisms $\Psi = (\varphi^1, \varphi^2)$ such that[14]

1. φ^1 operates on the restricted problem $(I,\ S^1, P^1, q|S^1, \succ |S^1, h)$ whose primitives are the set of all agents, a subset of all objects available for assignment in the first round (defined by the application), and the preferences and priorities over available objects; and[15]

2. φ^2 operates on the restricted problem $(I^2,\ S^2, P^2, q^2, \succ |S^2, h)$ whose primitives are the set of all agents (without property rights) who are unassigned in the first round, the set of all objects available for assignment in the second round (defined by the application), the preferences, and priorities over

[13]In the sequel we provide descriptions of the (agent-proposing) DA and the serial dictatorship mechanisms. We refer the reader to the extant literature for the descriptions of the remaining mechanisms.

[14]Alternatively, we can define the sequential assignment system by introducing q^1 instead of S^1 and S^2 where $q_s^1 = q_s$ for the schools available in step 1 and $q_s^1 = 0$ for the schools available in step 2. One can also think allowing agents with priority rights to apply in the second period will generalize the problem. However, that modification will not capture the features of the TAP which is the only real life example that we know for a sequential assignment system where some agents have property rights over the objects.

[15]The notations $q|S^1$ and $\succ |S^1$ respectively denote the restrictions of q and $\succ$ to the set of objects in S^1. Here, $P^1 = (P_i^1)_{i \in I}$ is the preference profile over the available objects in step 1. Similarly, $P^2 = (P_i^2)_{i \in I^2}$ is the preference profile over the available objects in step 2.

available objects. More precisely,

$$
\begin{aligned}
I^2 &= \{i : \varphi_i^1(I, S^1, P^1, q|S^1, \succ|S^1, h) = \emptyset and h(i) = \emptyset\}, \\
S^2 &= S \setminus S^1, \\
q_s^2 &= q_s - |\{i : h(i) = s and \varphi_i^1(I, S^1, P^1, q|S^1, \succ|S^1, h) = \emptyset\}| \ \forall s \in S^2.
\end{aligned}
$$

It is important to note that since the number of available copies of each object in the second round depends on the assignment in round 1, the problem in the second round (including the participating agents as well as available objects) is "endogenously" determined through the assignments made in the first round. Then, the assignment of agent i for a problem under system Ψ is defined formally as:

$$
\Psi_i(I, S^1, S^2, P^1, P^2, q, \succ, h) = \begin{cases} \varphi_i^1(I, S^1, P^1, q|S^1, \succ|S^1, h) if \varphi_i^1(I, S^1, P^1, q|S^1, \succ|S^1, h) \neq \emptyset, \\ h(i) if \varphi_i^1(I, S^1, P^1, q|S^1, \succ|S^1, h) = \emptyset and h(i) \neq \emptyset, \\ \varphi_i^2(I^2, S^2, P^2, q^2, \succ|S^2, h) otherwise. \end{cases}
$$

A system is **straightforward** if no agent ever gains by ranking available objects non-truthfully in any round she participates in. Formally, for every $i \in I$, every pair (P_i', P_i'') and every P, we have

$$
\Psi_i(I, S^1, S^2, P|S^1, P|(S^2, I^2), q, \succ, h) R_i \Psi_i(I, S^1, S^2, (P_i', P_{-i})|S^1, (P_i'', P_{-i})|(S^2, I^{2'}), q, \succ, h),
$$

where $I^{2'}$ is the corresponding set of agents available in round 2 for preference profile (P_i', P_{-i}).

A system is **non-wasteful {fair} [individually rational]** if for every initial problem P, $\Psi(I, S^1, S^2, P|S^1, P|(S^2, I^2), q, \succ, h)$is a non-wasteful {fair} [individually rational] matching.[16]

We say that $\widetilde{\succ}$ is an improvement in the priorities for agent $i \in I$ if (1) $i \succ_s j \Longrightarrow i \widetilde{\succ}_s j$ for all $s \in S$, (2) there exists at least one agent j' and school s' such that $j \succ_s i \widetilde{\succ}_s j$, and (3) $k \succ_s l \Longleftrightarrow k \widetilde{\succ}_s l$ for all $s \in S$ and $k, l \in I \backslash \{i\}$. A system Ψ **respects improvements in the priorities** if, if $\widetilde{\succ}$ is an improvement in the priorities for agent $i \in I$, then for any $i \in I$ we have

$$
\Psi_i(I, S^1, S^2, P|S^1, P|(S^2, I^{2'}), q, \widetilde{\succ}, h) R_i \Psi_i(I, S^1, S^2, P|S^1, P|(S^2, I^2), q, \succ, h),
$$

[16] Note that this definition ignores potential strategic behavior of agents in either of the two rounds. This will cause no loss of generality as our analysis will be focusing on straightforward systems.

where $I^{2'}$ is the corresponding set of agents available in Round 2 for priority order $\widetilde{\succ}$.

In the rest of the paper, whenever there is no ambiguity, we fix the set of agents, objects, quotas, priority orders, and the ownership profile, and represent the outcome of a system for a given problem by $\Psi(P^1, P^2)$. In line with the two real-life applications, we refer to objects as schools in the rest of the paper.

3 Two Applications

3.1 School Choice Problem with Exam and Regular Schools (SCPwEXRS)

A **school choice problem with exam and regular schools**, or a problem for short, consists of[17]

1. A set of schools $S = \{s_1, s_2, ..., s_m\}$. S is composed of two disjoint sets, i.e. exam and regular schools. Let S^e be the set of exam schools and S^r be the set of regular schools and $S = S^e \cup S^r$.

2. A capacity vector $q = (q_s)_{s \in S}$ where q_s is the number of available seats in $s \in S$.

3. A set of students $I = \{i_1, i_2, ..., i_n\}$.

4. A preference profile $P = (P_i)_{i \in I}$ where P_i is the strict preference of i over $S \cup \emptyset$.

5. A priority order $\succ = (\succ_s)_{s \in S}$ where $\succ_s$ is the strict priority order of students in I for school s.

6. An ownership profile $h = (h(i))_{i \in I}$ where $h(i)$ is the school for which agent i has the property right.

All the available seats in both types of schools are social endowments. Therefore, $h(i) = \emptyset$ for all $i \in I$.[18] Let $c(i)$ be the test score of applicant $i \in I$ and c be the test score profile of all applicants, $c = (c(i))_{i \in I}$.[19] In the school choice problem with exam and regular schools, only the exam schools rank the students based on their exam score. That is, for each $s \in S^e$, $i \succ_s j$ if and only if $c(i) > c(j)$. On the other hand, the regular schools use some predetermined exogenous rules (proximity, sibling status) to rank students. All exam schools rank students in the same order. However, the priorities of students for any two different regular schools need not be the same. Let $\succ$ be the set of all possible priority profiles in this environment. Then for any $\succ' \in \succ$, $\succ'_s = \succ'_{s'}$ for all $s, s' \in S^e$.

[17] We are using a similar notation as that of Balinski and Sonmez (1999).
[18] Alternatively, one can define the problem as a 5-tuple by excluding h.
[19] Here the test score profile is an exogenous rule that is used to determine the priority order.

The current system used in Boston is a **serial dictatorship followed by deferred acceptance mechanism** (SD-DA) and works as follows:

Round 1:

- Only exam schools, S^e, are available for assignment in this round and all students can participate. Students submit their preferences over the set S^e and $\emptyset$. Let $P^1 = (P_i^1)_{i \in I}$ be the list of submitted preferences in round 1. Therefore, the problem considered in round 1 is $(I, S^e, P^1, q|S^e, \succ |S^e, h)$.

- The serial dictatorship mechanism is applied to the problem $(I, S^e, P^1, q|S^e, \succ |S^e, h)$: The agent with the highest score is assigned to his top choice in the list he submitted, the next agent is assigned to his top choice among the remaining schools, and so on.

- Let μ_1 denote the assignment in round 1.

Round 2:

- The problem considered in round 2 is $(I^2, S^2, P^2, q^2, \succ |(S^2, I^2), h)$. I^2, S^2 and q^2 are calculated as described in Section **??**. Note that $S^2 = S^r$, $q^2 = (q_s)_{s \in S^r}$ and all the unassigned students in the first round participate.

- The student-proposing deferred acceptance mechanism is used in the placement process:

  - Each agent $i \in I^2$ applies to the top ranked school in P_i^2. Each school $s \in S^2$ tentatively accepts all best offers up to its quota q_s^2 according to its priority order. The rest are rejected.
  - In general: each agent $i \in I^2$ applies to the highest-ranked school in P_i^2 that has not rejected him yet. Each school that holds tentatively accepted offers or receives new offers in this round tentatively accepts all best acceptable offers, among the new and previously held ones, up to its quota according to its priority order. The rest are rejected.
  - The algorithm terminates when there are no more rejections.

- Let μ_2 be the final assignment in round 2.

The placement of agent $i \in I$ induced by the SD-DA is:

$$\mu(i) = \begin{cases} \mu_1(i) if \mu_1(i) \neq \emptyset, \\ \mu_2(i) otherwise. \end{cases}$$

3.2 Teacher Assignment Problem (TAP)

A **teacher assignment problem**, or a problem for short, consists of

1. A set of schools $S = \{s_1, s_2, ..., s_m\}$. S is composed of two disjoint subsets, i.e. contractual and tenured schools. Let S^c be the set of contractual schools and S^t be the set of tenured schools and $S = S^c \cup S^t$.

2. A capacity vector $q = (q_s)_{s \in S}$ where q_s is the number of available seats in $s \in S$.

3. A set of applicants $I = \{i_1, i_2, ..., i_n\}$. I is composed of two disjoint subsets, i.e. existing teachers and new graduates. Let I^e be the set of existing teachers and I^n be the set of new graduates and $I = I^e \cup I^n$.

4. A preference profile $P = (P_i)_{i \in I}$ where P_i is the strict preference of i over $S \cup \emptyset$.[20]

5. A priority order $\succ = (\succ_s)_{s \in S}$ where $\succ_s$ is the strict priority order of applicants in I for school s.

6. An ownership profile $h = (h(i))_{i \in I}$ where $h(i)$ is the school for which agent i has property rights.

Each $i \in I^e$ has property rights over a school in $S^c,_{i \in I^e} 1(\ h(i) = s) = q_s$ for all $s \in S^c$. For each new graduate $i \in I^n$ $h(i) = \emptyset$. The number of available seats in $s \in S^c$ is $q_s = |h^{-1}(s)|$. All the available seats in tenured schools are social endowments.

The strict priority order of applicants in I for each school s, $\succ_s$, is determined according to the centralized test score of each agent and the property rights. Each tenured school $s \in S^r$ ranks applicants based on their test scores: $i \succ_s j$ if and only if $c(i) > c(j)$. Each existing teacher currently working in a contractual school $s \in S^c$ is given the right to keep his position unless he is assigned to a better school. That is, each contractual school $s \in S^c$ ranks its current teachers at the top of its priority order. Each contractual school $s \in S^c$ considers each existing teacher working in another contractual school to be unacceptable. All the new graduates are ranked based on their test scores. The priority order of each contractual school $s \in S^c$ is:

- For all $i, j \in I$ such that $h(i) = s$, $h(j) \neq s$ then $i \succ_s j$

- For all i, $j \in I$ such that $h(i) = h(j)$, $i \succ_s j$ if and only if $c(i) > c(j)$

[20] All existing teachers are assumed to prefer their current position to $\emptyset$.

- For each $s \in S^c$ and all $i \in I^e$ such that $h(i) \neq s$ then $\emptyset \succ_s i$

- For each $s \in S^c$ and all $i \in I^n$ and $j \in I^e$ such that $h(j) \neq s$, $i \succ_s j$.

It is worth mentioning that if there exist two new graduates $i, j \in I^n$ and two schools $s, s' \in S$ such that $j \succ_s j$ and $i \succ_{s'} j$, then $\succ$ cannot be a possible priority profile in this environment, i.e. $\succ \notin \succ$.

As one can notice, in the TAP the priority order can be constructed by using the test scores and the ownership profile. Alternatively, we can define the TAP as (I, S, P, q, c, h). To be consistent with the general framework we define a problem as $(I, S, P, q, \succ, h)$.

The system that was in use in Turkey until very recently is the **two-round serial dictatorship mechanism** (TRSD) and works as follows:

Round 1:

- Only tenured schools, S^t, are available for assignment in this round and all teachers can participate. Teachers submit their preferences over the set S^t and $\emptyset$. Let $P^1 = (P_i^1)_{i \in I}$ be the list of submitted preference in round 1. Therefore, the problem considered in round 1 is $(I, S^t, P^1, q|S^t, \succ|S^t, h)$.[21]

- The serial dictatorship mechanism is applied to the problem $(I, S^t, P^1, q|S^t, \succ|S^t, h)$: The agent with the highest score is assigned to his top choice in the list he submitted, the next agent is assigned to his top choice among the remaining schools, and so on.

- Let μ_1 denote the assignment in round 1.

Round 2:

- The problem considered in round 2 is $(I^2, S^2, P^2, q^2, \succ|(S^2, I^2), h)$. I^2, S^2 and q^2 are calculated as described in Section **??**. Note that $S^2 = S^c$ and only the unassigned new graduates participate.[22]

- The serial dictatorship mechanism is used in the assignment process: The agent with the highest test score is assigned to his top choice in the list he submitted. The number of available seats in that school is updated and if it falls to zero that school is removed. The agent with the second highest test score is assigned to his top choice among the remaining schools, and so on.

[21] Since only the tenured schools are considered in this round, the priority order for all available schools are the same and it is equivalent to the order of test scores.

[22] Since only the new graduates can participate in this round, each school $s \in S^2$ ranks the agents in I^2 based on test scores.

- Let μ_2 be the final assignment in round 2.

The placement of agent $i \in I$ induced by the TRSD is:

$$\mu(i) = \begin{cases} \mu_1(i) if \mu_1(i) \neq \emptyset \\ h(i) if \mu_1(i) = \emptyset and h(i) \neq \emptyset \\ \mu_2(i) otherwise. \end{cases}$$

4 Deficiencies of General Sequential Systems

We next show that sequential systems, regardless of the specific mechanisms used in each round, may be inherently flawed. To this end, we offer some impossibility results. In Theorem **??**, we show that any straightforward system is wasteful and therefore not efficient.

Theorem 1 *There does not exist a straightforward and non-wasteful system.*

Proof. We argue by contradiction. Suppose $\Psi(\varphi^1, \varphi^2)$ is straightforward and non-wasteful.To show that the result does not depend on the choice of the ownership structure, we consider two different cases. In the first case, all schools are social endowment. In the second case, we allow some schools to be owned.

Case 1: There are three schools $S = \{s_1, s_2, s_3\}$ with one available seat and two agents $I = \{i_1, i_2\}$. Let $S^1 = \{s_2, s_3\}$ and $h(i_1) = h(i_2) = \emptyset$. The priority structure of each school is the same and given as $i_1 \succ_s i_2 \succ_s \emptyset$ for all $s \in S$. Let the true preferences be as follows:

$$s_2 P_{i_1} s_3 P_{i_1} s_1 P_{i_1} \emptyset$$

$$s_1 P_{i_2} s_2 P_{i_2} s_3 P_{i_2} \emptyset$$

There is only one non-wasteful matching $\mu_1 = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_2 & i_1 & \emptyset \end{pmatrix}$. In the first round, we have the problem $(I, S^1, P^1, q|S^1, \succ |S^1, h)$ where $s_2 P^1_{i_1} s_3 P^1_{i_1} \emptyset$ and $s_2 P^1_{i_2} s_3 P^1_{i_2} \emptyset$. The matching selected in round 1 is $\varphi^1_{i_1}(I, S^1, P^1, q|S^1, \succ |S^1, h) = s_2$, $\varphi^1_{i_2}(I, S^1, P^1, q|S^1, \succ |S^1, h) = \emptyset$.

Now consider the following preference profile:

$$s_2 \overline{P}_{i_1} s_3 \overline{P}_{i_1} s_1 \overline{P}_{i_1} \emptyset$$

$$s_2 \overline{P}_{i_2} s_3 \overline{P}_{i_2} s_1 \overline{P}_{i_2} \emptyset$$

There are two non-wasteful matchings $\mu_2 = \begin{pmatrix} s_1 & s_2 & s_3 \\ \emptyset & i_1 & i_2 \end{pmatrix}$ and $\mu_3 = \begin{pmatrix} s_1 & s_2 & s_3 \\ \emptyset & i_2 & i_1 \end{pmatrix}$. In the first round, we have the following problem $(I, S^1, \overline{P}^1, q|S^1, \succ |S^1, h)$ where $s_2 \overline{P}^1_{i_1} s_3 \overline{P}^1_{i_1} \emptyset$ and $s_2 \overline{P}^1_{i_2} s_3 \overline{P}^1_{i_2} \emptyset$. Note that $(I, S^1, \overline{P}^1, q|S^1, \succ |S^1, h) = (I, S^1, P^1, q|S^1, \succ |S^1, h)$ but $\varphi^1(I, S^1, \overline{P}^1, q|S^1, \succ |S^1, h) \neq \varphi^1(I, S^1, P^1, q|S^1, \succ |S^1, h)$. A contradiction.

Case 2: Consider the same example. We only change the example by giving the property rights of school s_1 to i_1. That is, $h(i_1) = s_1$ and $h(i_2) = \emptyset$. The non-wasteful matchings for both preference profiles will be the exact matching that we get in Case 1. Therefore, we have a contradiction in this case too. ■

Since Pareto efficiency implies non-wastefulness, Theorem **??** has an immediate corollary.

Corollary 1 *There does not exist a straightforward and Pareto efficient system.*

Note that a system that always select the null matching at which all agents are unassigned is fair and respects improvements. In Theorem **??** we show that it is not possible to construct a straightforward and fair system by using non-wasteful mechanisms in each round.

Theorem 2 *Let $\Psi = (\varphi^1, \varphi^2)$ be a system. If φ^1 and φ^2 are non-wasteful and Ψ is straightforward, then Ψ cannot be fair.*

Proof. As in the proof of Theorem **??**, we consider two different cases. In the first case, all schools are social endowments. In the second case, we allow some schools to be owned by agents.

Case 1: There are three schools $S = \{s_1, s_2, s_3\}$ with one available seat and three agents $I = \{i_1, i_2, i_3\}$. Let $S^1 = \{s_2, s_3\}$, and $h(i_1) = h(i_2) = h(i_3) = \emptyset$. The priority structure of each school in S is the same and given as $i_1 \succ_s i_2 \succ_s i_3 \succ_s \emptyset$ for all $s \in S$. Let the true preferences be as follows:

$$s_2 P_{i_1} s_3 P_{i_1} s_1 P_{i_1} \emptyset$$

$$s_1 P_{i_2} s_2 P_{i_2} s_3 P_{i_2} \emptyset$$

$$s_1 P_{i_3} s_2 P_{i_3} s_3 P_{i_3} \emptyset$$

When all agents act truthfully (straightforwardly), any fair system composed of two non-wasteful mechanisms will select: $\mu_1 = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_2 & i_1 & i_3 \end{pmatrix}$. In the first round, we have the following problem $(I, S^1, P^1, q|S^1, \succ$

$|S^1, h)$ where $s_2 P^1_{i_1} s_3 P^1_{i_1} \emptyset$, $s_2 P^1_{i_2} s_3 P^1_{i_2} \emptyset$ and $s_2 P^1_{i_3} s_3 P^1_{i_3} \emptyset$. The matching selected in round 1 is $\varphi^1_{i_1}(I, S^1, P^1, q|S^1, \succ |S^1, h) = s_2$, $\varphi^1_{i_2}(I, S^1, P^1, q|S^1, \succ |S^1, h) = \emptyset$ and $\varphi^1_{i_3}(I, S^1, P^1, q|S^1, \succ |S^1, h) = s_3$.

Now consider the following preference profile:

$$s_2 \overline{P}_{i_1} s_3 \overline{P}_{i_1} s_1 \overline{P}_{i_1} \emptyset$$

$$s_2 \overline{P}_{i_2} s_3 \overline{P}_{i_2} s_1 \overline{P}_{i_2} \emptyset$$

$$s_1 \overline{P}_{i_3} s_2 \overline{P}_{i_3} s_3 \overline{P}_{i_3} \emptyset$$

When all agents act truthfully (straightforwardly), any fair system composed of two non-wasteful mechanisms will select: $\mu_2 = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_3 & i_1 & i_2 \end{pmatrix}$. In the first round, we have the following problem $(I, S^1, \overline{P}^1, q|S^1, \succ |S^1, h)$ where $s_2 \overline{P}^1_{i_1} s_3 \overline{P}^1_{i_1} \emptyset$, $s_2 \overline{P}^1_{i_2} s_3 \overline{P}^1_{i_2} \emptyset$ and $s_2 \overline{P}^1_{i_3} s_3 \overline{P}^1_{i_3} \emptyset$. Note that $(I, S^1, \overline{P}^1, q|S^1, \succ |S^1, h) = (I, S^1, P^1, q|S^1, \succ |S^1, h)$ but $\varphi^1(I, S^1, \overline{P}^1, q|S^1, \succ |S^1, h) \neq \varphi^1(I, S^1, P^1, q|S^1, \succ |S^1, h)$. A contradiction.

Case 2: Consider the same example. We change the example only by giving the property rights of school s_1 to i_1, keeping everything else the same. That is, $h(i_1) = s_1$. Given that the problems observed in the first round are the same as in Case 1, we will observe the same problem in the second round as in the Case 1. Hence, the same matchings will be selected in both preference profiles as in tCase 1 and we again have a contradiction. ■

Theorem 3 states that a straightforward system cannot respect improvements if it employs non-wasteful and fair mechanism in the first round and non-wasteful mechanism in the second round .

Theorem 3 *Let $\Psi = (\varphi^1, \varphi^2)$ be a straightforward system. If φ^1 is non-wasteful and fair, and φ^2 is non-wasteful, then Ψ does not respect improvements in the priority order.*

Proof. We argue by contradiction. Suppose $\Psi(\varphi^1, \varphi^2)$ is straightforward, φ^1 is non-wasteful and fair, and φ^2 is non-wasteful and Ψ respects improvements. We consider two different cases. In the first case, all schools are social endowments. In the second case, we allow some schools to be owned.

Case 1: There are three schools $S = \{s_1, s_2, s_3\}$ with one available seat and three agents $I = \{i_1, i_2, i_3\}$. Let $S^1 = \{s_2, s_3\}$, and $h(i_1) = h(i_2) = h(i_3) = \emptyset$. The priority structure of each school in S is the same and given as $i_1 \succ_s i_2 \succ_s i_3 \succ_s \emptyset$ for all $s \in S$. Let the true preferences be as follows:

$$s_2 P_{i_1} s_3 P_{i_1} s_1 P_{i_1} \emptyset$$

$$s_1 P_{i_2} s_2 P_{i_2} s_3 P_{i_2} \emptyset$$

$$s_1 P_{i_3} s_2 P_{i_3} s_3 P_{i_3} \emptyset$$

When all agents act truthfully (straightforwardly), any straightforward system with mechanisms as described in the statement will select: $\mu_1 = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_3 & i_1 & i_2 \end{pmatrix}$.

Now consider the following priority order: $i_1 \succ'_s i_3 \succ'_s i_2 \succ'_s \emptyset$ for all $s \in S$. When all agents act truthfully, any straightforward system with mechanisms as described in the statement will select: $\mu_2 = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_2 & i_1 & i_3 \end{pmatrix}$. Since $\mu_1(i_3) P_{i_3} \mu_2(i_3)$, Ψ does not respect the improvement of i_3.

Case 2: Consider the same example. We change the example only by giving the property rights of school s_1 to i_1, keeping everything else the same. That is, $h(i_1) = s_1$. Given that the problems observed in the first round are the same as in Case 1, we will observe the same problem in the second round as in Case 1. Hence, the same matchings will be selected in both preference profiles as in Case 1 and we have a contradiction. ■

In Theorem **??**, we show that a straightforward system cannot be obtained by using non-wasteful mechanisms in both rounds and an individually rational mechanism in the first round.

Theorem 4 *Let $\Psi = (\varphi^1, \varphi^2)$ be a system. If φ^1 is non-wasteful and individually rational and φ^2 is non-wasteful, then Ψ fails to be straightforward.*

Proof. We consider two different cases. In the first case, all schools are social endowments. In the second case, we allow some schools to be owned.

Case 1: There are three schools $S = \{s_1, s_2, s_3\}$ with one available seat and two agents $I = \{i_1, i_2\}$. Let $S^1 = \{s_2, s_3\}$, and $h(i_1) = h(i_2) = \emptyset$. The priority structure of each school is the same and given as $i_1 \succ_s i_2 \succ_s \emptyset$ for all $s \in S$. Let the true preferences be as follows:

$$s_2 P_{i_1} s_3 P_{i_1} s_1 P_{i_1} \emptyset$$

$$s_1 P_{i_2} s_2 P_{i_2} s_3 P_{i_2} \emptyset$$

In the first round, if both agents act truthfully and submit their ranking lists by keeping the relative order of available schools and $\emptyset$ in P then there are two non-wasteful and individually rational matchings: $\mu_1^1 = \begin{pmatrix} s_2 & s_3 \\ i_1 & i_2 \end{pmatrix}$ and $\mu_1^2 = \begin{pmatrix} s_2 & s_3 \\ i_2 & i_1 \end{pmatrix}$. No matter which one of these two matchings is selected in the first round, none of the agents can participate the second round and s_1 is available in the second round. Therefore, the unique non-wasteful matching selected in the second period is $\mu_2^1 = \mu_2^2 = \begin{pmatrix} s_1 \\ \emptyset \end{pmatrix}$. That is, any system satisfying conditions mentioned in the statement of Theorem **??** assigns i_2 to either s_2 or s_3. Let $\widetilde{\Psi}$ be a system selecting (μ_1^1, μ_2^1) and $\overline{\Psi}$ be a system selecting (μ_1^2, μ_2^2). That is, the outcome of $\widetilde{\Psi}$ is $\mu^1 = \begin{pmatrix} s_1 & s_2 & s_3 \\ \emptyset & i_1 & i_2 \end{pmatrix}$ and the outcome selected by $\overline{\Psi}$ is $\mu^2 = \begin{pmatrix} s_1 & s_2 & s_3 \\ \emptyset & i_2 & i_1 \end{pmatrix}$.

Suppose i_2 submits the following preference list in the first round: $\emptyset P'_{i_2} s_2 P'_{i_2} s_3$. There is a unique individually rational and non-wasteful matching in round 1: $\mu'_1 = \begin{pmatrix} s_2 & s_3 \\ i_1 & \emptyset \end{pmatrix}$, and both $\widetilde{\Psi}$ and $\overline{\Psi}$ select μ'_1. Based on the matching selected in the first round, i_2 can participate in the second round and s_1 is available in the second round. When i_2 submits $P''_{i_2} : s_1 P''_{i_2} \emptyset$, then there is a unique non-wasteful matching: $\mu'_2 = \begin{pmatrix} s_1 \\ i_2 \end{pmatrix}$. That is, the pair (P'_{i_1}, P''_{i_1}) is a profitable deviation for i_2 under $\widetilde{\Psi}$ and $\overline{\Psi}$. Note that P''_2 is i_1's true relative order over the available schools in the second round.

Case 2: Consider the same example. We change the example only by giving the property rights of school s_1 to i_1. That is, $h(i_1) = s_1$ and $h(i_2) = \emptyset$. Since the set of available schools in round 1 is not changed, then in the first round in any non-wasteful matching i_1 will be assigned to another school and he will give up his property rights for s_1. Therefore, s_1 will be socially endowed in the second rounds as in Case 1. One can follow the steps in Case 1 and show that the impossibility result is robust to the ownership structure, i.e. whether all schools are socially endowed or not. ■

Recall that when defining a system, we impose certain restrictions. For instance, the unfilled seats of $s \in S^1$ are not available in round 2, or if an agent with an endowment is not assigned to a school in S^1, he cannot participate in the second round. We highlight that these constraints, which we adopt for ease of exposition, do not play any role in obtaining the above results. Indeed, Theorems **??**, **??**, and **??** hold in the absence of these restrictions.

Remark 1 *Theorems* **??**, **??**, *and* **??** *hold in the following cases:*

(1) The schools in S^1 with unfilled seats are available in the second round.

(2) Unassigned agents with endowments are allowed to participate in the second round.

(3) Only particular slots of the schools are available and not all students are allowed to participate in the first round.

(4) Only particular slots of the schools are owned.

If we use a non-wasteful mechanism in the first round, then Theorem **??** holds in the absence of these restrictions.

Remark 2 *When a non-wasteful mechanism is used in the first round Theorem* **??** *holds in the following cases:*

(1) The schools in S^1 with unfilled seats are available in the second round.

(2) Unassigned agents with endowments are allowed to participate in the second round.

(3) Only particular slots of the schools are available and not all students are allowed to participate in the first round.

(4) Only particular slots of the schools are owned.

In TAP and SCPwEXRS, different combinations of DA and SD mechanisms are used in different rounds. Both DA and SD are non-wasteful and individually rational. Hence, Theorem **??** has the following immediate corollary for the two applications we have considered.

Corollary 2 *SD-DA used in SCPwEXRS and TRSD used in TAP are vulnerable to manipulation.*

We focus on the case where agents reports their roundwise true preferences over the available schools. In Proposition **??**, we point out the deficiencies of both systems in practice.

Proposition 1 *SD-DA used in SCPwEXRS and TRSD used in TAP are wasteful, not fair, and do not respect improvements in the priority order (test scores).*

Proof. The proof follows from the proofs of Theorems **??**, **??**, and **??**. ■

In TAP, the order in which the assignments to the tenured and cntractual positions are made cannot be changed since, in order to fill the vacated contractual positions, the assignment of contractual teachers to tenured positions must be handled first. On the other hand, in SCPwEXRS, the assignment order can be changed by first assigning students to the regular schools and then assigning the remaining students to the exam schools. One can then wonder whether the deficiencies of the sequential system used in school choice are a consequence of considering exam schools before the regular schools. It follows as a corollary that if we first consider the regular schools and then the exam schools, unfortunately the same deficiencies arise.

Corollary 3 *DA-SD in SCPwEXRS is manipulable, wasteful, not fair, leads to avoidable welfare loss and does not respect improvements in the priority order.*

In Appendix **??** we present two examples to illustrate how the SD-DA and TRSD mechanisms fail to satisfy the desired properties.

5 Equilibrium Analysis of the Preference Revelation Games

In Section **??**, we have shown that the systems in Turkey and the U.S. are not straightforward. We further argue that it may not be difficult for agents to identify strategies that allow them to manipulate these systems. In this section, we first investigate possible ways of manipulation under the sequential systems. Then, we turn to analyzing the properties of the preference revelation games associated with the current systems. Since both systems are composed of two rounds, we consider the subgame perfect Nash equilibrium (SPNE) as the main solution concept. We analyze the games under complete information of payoffs, available strategies, and priorities. Agents are assumed to play simultaneously, and the outcome of the first round is announced publicly.

We first show that, in the general setting, if an agent can gain from misreporting, then this implies that he will also be weakly better off by declaring all available schools unacceptable in the first round and reporting his true relative-preferences over the available schools in the second round.

Denote i's true relative preference over the available schools in round $t \in \{1, 2\}$ including $\emptyset$ with P_i^t.

Proposition 2 *Let $\Psi = (\varphi^1, \varphi^2)$ be a system such that both φ^1 and φ^2 are strategy-proof and individually rational and φ^1 is non-wasteful and weak population monotonic. If there exists a preference pair (Q_i^1, Q_i^2) such that $\Psi_i((Q_i^1, P_{-i}^1), (Q_i^2, P_{-i}^2)) P_i \Psi_i(P^1, P^2)$ then $\Psi_i((\widetilde{Q}_i^1, P_{-i}^1), (\widetilde{Q}_i^2, P_{-i}^2))\ R_i\ \Psi_i((Q_i^1, P_{-i}^1), (Q_i^2, P_{-i}^2))$ where $\widetilde{Q}_i^1 = \emptyset\ \widetilde{Q}_i^1\ x$ for all $x \in S^1$ and $\widetilde{Q}_i^2 = P_i^2$.*

Proof. Suppose $\Psi_i((Q_i^1, P_{-i}^1),(Q_i^2, P_{-i}^2))P_i\Psi_i(P^1, P^2)$ and $\Psi_i((Q_i^1, P_{-i}^1),(Q_i^2, P_{-i}^2))P_i\Psi_i((\widetilde{Q}_i^1, P_{-i}^1),(\widetilde{Q}_i^2, P_{-i}^2))$. By individual rationality, $\Psi_i(P^1, P^2)R_i\emptyset$. Hence, $\Psi_i((Q_i^1, P_{-i}^1),(Q_i^2, P_{-i}^2))P_i\emptyset$ and $\Psi_i((Q_i^1, P_{-i}^1),(Q_i^2, P_{-i}^2)) \in S$. We consider two possible cases.

Case 1: $\Psi_i((Q_i^1, P_{-i}^1),(Q_i^2, P_{-i}^2)) \in S^1$. In this case, $\varphi_i^1(Q_i^1, P_{-i}^1) = \Psi_i((Q_i^1, P_{-i}^1),(Q_i^2, P_{-i}^2))$ and i does not participate in the second round. Since φ^1 is strategy-proof and $\varphi_i^1(Q_i^1, P_{-i}^1)P_i\emptyset$, $\varphi_i^1(P^1)R_i\varphi_i^1(Q_i^1, P_{-i}^1)P_i\emptyset$. Hence, $\Psi_i(P^1, P^2) = \varphi_i^1(P^1)R_i\Psi_i((Q_i^1, P_{-i}^1),(Q_i^2, P_{-i}^2))$. This is a contradiction.

Case 2: $\Psi_i((Q_i^1, P_{-i}^1),(Q_i^2, P_{-i}^2)) \in S^2$. In this case, $\varphi_i^2(Q_i^2, P_{-i}^2) = \Psi_i((Q_i^1, P_{-i}^1),(Q_i^2, P_{-i}^2))$ and $\varphi_i^1(Q_i^1, P_{-i}^1) = \emptyset$. By individual rationality, i is unassigned when he submits $\widetilde{Q}_i^1 = \emptyset\widetilde{Q}_i^1 x$ for all $x \in S^1$. If $\varphi_j^1(Q_i^1, P_{-i}^1) \in S^1$ then $\varphi_j^1(\widetilde{Q}_i^1, P_{-i}^1) \in S^1$ due to individual rationality ($\varphi_j^1(Q_i^1, P_{-i}^1)P_j\emptyset$) and population monotonicity ($\varphi_j^1(\widetilde{Q}_i^1, P_{-i}^1)R_j\varphi_j^1(Q_i^1, P_{-i}^1)$). Moreover, if $\varphi_j^1(Q_i^1, P_{-i}^1) \neq \varphi_j^1(\widetilde{Q}_i^1, P_{-i}^1)$ then $\varphi_j^1(\widetilde{Q}_i^1, P_{-i}^1)$ should have filled all its available seats in matching $\varphi^1(Q_i^1, P_{-i}^1)$. Otherwise non-wastefulness of φ^1 is violated. That is, only the agents who are assigned to a school in $\varphi^1(Q_i^1, P_{-i}^1)$ become better off in $\varphi^1(\widetilde{Q}_i^1, P_{-i}^1)$. Hence, the same set of students is assigned to a school in S^1 and each school fills the same number of seats in $\varphi^1(Q_i^1, P_{-i}^1)$ and $\varphi^1(\widetilde{Q}_i^1, P_{-i}^1)$. In other words, the same set of agents will participate in the second round and the quotas of each school in S^2 will be the same when i submits Q_i^1 and $\widetilde{Q}_i^1$. By strategy-proofness, i cannot get a better school than $\varphi_i^2(\widetilde{Q}_i^2 = P_i^2, P_{-i}^2)$ in $\varphi^2(Q_i^2, P_{-i}^2)$. Therefore, $\Psi_i((\widetilde{Q}_i^1, P_{-i}^1),(\widetilde{Q}_i^2, P_{-i}^2))R_i\Psi_i((Q_i^1, P_{-i}^1),(Q_i^2, P_{-i}^2))$. A contradiction. ■

Note that the mechanisms used in the first and second rounds of TRSD, SD-DA and DA-SD satisfy the conditions mentioned in Proposition **??**. Therefore, for a given problem, if an agent has a way of manipulating these sequential assignment systems, then she can equivalently extract all the benefits of that manipulation by ranking $\emptyset$ as his first choice in the first round and acting truthfully in the second round.

Corollary 4 *If an agent benefits from misreporting under TRSD in TAP or SD-DA (DA-SD) in SCP-wEXRS, then ranking $\emptyset$ as the first choice in the first round and acting truthfully in the second round extracts all benefits from manipulation.*

In Section **??**, we showed that TRSD is not straightforward. However, Proposition **??** shows that not all of the applicants can benefit from misreporting their preferences. In particular, existing teachers cannot benefit from misreporting, because when an existing tenant ranks $\emptyset$ as his first choice, he gets her current position, which is the worst outcome he receives under truth-telling.

Corollary 5 *Under TRSD, existing teachers cannot benefit from misreporting.*

In Proposition **??**, we show that if a system satisfies the conditions in Proposition **??**, then the only way to manipulate that system is to truncate the relative preferences over the available schools in Round 1.

Proposition 3 *Let $\Psi = (\varphi^1, \varphi^2)$ be a system such that both φ^1 and φ^2 are strategy-proof and φ^1 is individually rational, non-wasteful, and weak population monotonic. If whenever there exists a preference pair (Q_i^1, Q_i^2) such that $\Psi_i((Q_i^1, P_{-i}^1), (Q_i^2, P_{-i}^2)) P_i \Psi_i(P^1, P^2)$, then there exists a school $s \in S^1$ where $\emptyset\ Q_i^1\ s$ and $s\ P_i^1\ \emptyset$.*

Proof. By contradiction, we show that there does not exist a profitable deviation in which all the acceptable schools under the true preference are ranked above the being unassigned option in the reported preference list of the first round. We consider two cases: $\varphi_i^1(P^1) = \emptyset$ and $\varphi_i^1(P^1) \neq \emptyset$.

Case 1: By strategy-proofness, $\varphi_i^1(Q_i^1, P_{-i}^1) = \emptyset$. If $h(i) \neq \emptyset$ then $\Psi_i((Q_i^1, P_{-i}^1), (Q_i^2, P_{-i}^2)) = \Psi_i(P^1, P^2)$. If $h(i) = \emptyset$ then consider the preference profile $\widehat{Q}_i^1 : \emptyset\ \widehat{Q}_i^1\ s$ for all $s \in S^1$. By non-wastefulness and population monotonicity, the same set of students is assigned to schools in S^1 and each school fills the same number of seats under $\varphi^1(\widehat{Q}_i^1, P_{-i}^1)$, $\varphi^1(P^1)$ and $\varphi^1(Q_i^1, P_{-i}^1)$. Then the same set of agents will participate in the second round and the quotas of each school in S^2 will be the same when i submits P_i^1, Q_i^1 and $\widehat{Q}_i^1$. Since φ^2 is strategy-proof, i cannot be assigned to a better school than $\Psi_i(P^1, P^2) = \varphi_i^2(P^2)$.

Case 2: By strategy-proofness, if $\varphi_i^1(Q_i^1, P_{-i}^1) = \Psi_i((Q_i^1, P_{-i}^1), (Q_i^2, P_{-i}^2))$, then $\varphi_i^1(P^1) = \Psi_i(P^1, P^2)\ R_i\ \Psi_i((Q_i^1, P_{-i}^1), (Q_i^2, P_{-i}^2))$. That is, there does not exist a profitable deviation in which i is assigned a school in S^1. On the other hand, $\varphi_i^1(Q_i^1, P_{-i}^1) = \emptyset$ cannot be true due to the strategy-proofness of φ^1, i.e. P_i^1 would be a profitable deviation for someone whose real preference is Q_i^1 and that agent will get $\varphi_i^1(P^1)$. Note that $\varphi_i^1(P^1)\ P_i^1\ \emptyset$ since φ^1 is individually rational. By our construction $\varphi_i^1(P^1)\ Q_i^1\ \emptyset$. Hence, an agent with preference profile Q_i^1 can be better off by submitting P^1. ■

Remark 3 *Propositions **??** and **??** hold in the following cases:*

(1) The schools in S^1 with unfilled seats are available in the second round.

(2) Unassigned agents with endowment are allowed to participate in the second round.

Both the DA and SD mechanisms are strategy-proof, non-wasteful, population monotonic, and individually rational. Therefore, all three systems, TRSD, DA-SD, and SD-DA, satisfy the conditions mentioned in Proposition **??**. Hence, truncating the preference list over the schools in round 1 is the only way to manipulate TRSD in TAP or SD-DA (DA-SD) in SCPwEXRS.

Corollary 6 *If an agent benefits from misreporting under TRSD in TAP or SD-DA (DA-SD) in SCP-wEXRS, then that agent truncates his preference list of acceptable schools in round 1 by excluding at least one acceptable school.*

Now we are ready to start our equilibrium analysis. In the following theorems we provide general results on the SPNE analysis of systems.

Theorem 5 *Let $\Psi = (\varphi^1, \varphi^2)$ be a system such that*

- *φ^1 is individually rational, non-wasteful and either fair or [population monotonic and weakly non-bossy] and*
- *φ^2 is individually rational, non-wasteful and either fair or [monotonic, independent of irrelevant agents and weakly non-bossy].*

Every SPNE outcome of the preference revelation game associated with Ψ leads to a non-wasteful and individually rational matching.

Proof. Let $Q = (Q_i^1, Q_i^2)_{i \in I}$ be an SPNE profile and μ be the associated equilibrium outcome. First note that for any $i \in I$ we cannot have $\emptyset \succ_{\mu(i)} i$ because both φ^1 and φ^2 are individually rational. If μ is not individually rational, then there exists $i \in I$ such that $\emptyset \; P_i \; \mu(i)$. If $\emptyset \; P_i \; \mu(i)$, then submitting $P_i' = \emptyset \; P_i' x$ for all $x \in S^t$ in round $t \in \{1, 2\}$ is a profitable deviation for agent i. Therefore, Q cannot be SPNE profile, which is a contradiction.

Suppose μ is wasteful. Then, there exists $i \in I$ such that $sP_i\mu(i)$, $i \succ_s \emptyset$ and $|\mu^{-1}(s)| < q_s$. We consider two cases and we show that if μ is wasteful then i can profitably deviate.

Case 1: Suppose $s \in S^1$. Since φ^1 is non-wasteful, $\emptyset \; Q_i^1 \; s$. Consider following preference profile $P_i' = s \; P_i' \; \emptyset \; P_i' \; x$ for all $x \in S^1 \backslash \{s\}$. Denote $\varphi^1(P_i', Q_{-i}^1)$ by μ_1. By individual rationality, either $\mu_1(i) = s$ or $\mu_1(i) = \emptyset$. We consider two subcases:

φ^1 is individually rational, non-wasteful and fair. By the rural hospital theorem (Roth 1986) the same set of students is assigned to schools and each school fills the same number of seats at all fair, non-wasteful,

and individually rational matchings. Then consider the outcome of the sequential DA mechanism (McVitie and Wilson 1971) where student i applies after all students are tentatively assigned. Since DA is population monotonic, the number of students tentatively assigned to s before i's turn is less than q_s. When it is i's turn, he will be assigned to s. Hence, $\mu_1(i) \in S^1$ and this school is s, which is the only acceptable one in P_i'.

φ^1 *is individually rational, non-wasteful, population monotonic, and weakly non-bossy:* If $\mu_1(i) = s$ then (P_i', Q_i^2) is a profitable deviation for i. If $\mu_1(i) = \emptyset$, then $|\mu_1^{-1}(s)| = q_s$. Otherwise, μ_1 is wasteful. Let $\widetilde{I} = \{j \in I | \mu(j) \neq \mu_1(j) = s\}$. Since $|\mu_1^{-1}(s)| = q_s$ and $|\mu^{-1}(s)| < q_s$, $\widetilde{I} \neq \emptyset$. For all $j \in \widetilde{I}$ we have $\mu(j) Q_j^1 s Q_j^1 \emptyset$. Otherwise, φ^1 cannot be non-wasteful or individually rational. Now consider problem $(\overline{P}_i, Q_{-i}^1)$ where $\emptyset \overline{P}_i x$ for all $x \in S^1$. By non-bossiness and individual rationality, $\varphi^1(\overline{P}_i, Q_{-i}^1) = \mu_1$. When we consider problems $(\overline{P}_i, Q_{-i}^1)$ and (Q_i^1, Q_{-i}^1), all students should weakly prefer μ_1 to μ by population monotonicity. However, agents in $\widetilde{I}$ prefer μ to μ_1. A contradiction.

Case 2: Suppose $s \in S^2$. If $\mu(i) \in S^2 \cup \emptyset$, then by using the same steps in Case 1, one can show that (Q_i^1, P_i'') is a profitable deviation for i where $P_i'' = s\ P_i''\ \emptyset\ P_i'' x$ for all $x \in S^2 \setminus \{s\}$. If $\mu(i) \in S^1$ then we show that $(\overline{P}_i, P_i'')$ is a profitable deviation for i. Since φ^1 is individually rational and fair and/or population monotonic, if $\varphi_j^1(Q^1) \in S^1$ then $\varphi_j^1(\overline{P}_i, Q_{-i}^1) \in S^1$. Therefore, in the second round the set of agents is a subset of $I^2 \cup \{i\}$ and the set of available seats weakly increases compared to the case in which i plays Q_i^1. Let I_2', μ_2 and $\widetilde{q}^2$ respectively be the set of agents, the selected matching, and the quota vector in round 2 when i submits $(\overline{P}_i, P_i'')$. By individual rationality, $\mu_2(i)$ is either s or $\emptyset$. We show that $\mu_2(i)$ cannot be $\emptyset$. Suppose $\mu_2(i) = \emptyset$. Let $\widetilde{I}_2 = \{j \in I_2' | \mu(j) \neq \mu_2(j) = s\}$. Since φ^2 is non-wasteful, $|\mu_2^{-1}(s)| = q_s$, $\widetilde{I}_2 \neq \emptyset$ and $\mu(j)\ Q_j^2\ \mu_2(j)$ for all $j \in \widetilde{I}_2$. We consider two subcases:

φ^2 *is individually rational, non-wasteful and fair:* The students who participate in round 2 except i cannot fill all the available seats of s. Hence, the result follows from the same argument in Case 1.

φ^2 *is individually rational, non-wasteful, non-bossy, monotonic, and independent of irrelevant agents:* By non-bossiness and individual rationality, μ_2 will be selected when i submits $\widetilde{P}_i^2 = \emptyset\ \widetilde{P}_i^2\ x$ for all $x \in S^2$. Let μ_2' be the outcome of φ^2 when we consider only agents in $I_2' \setminus \{i\}$, keeping everything else the same. By monotonicity and independence of irrelevant agents, $\mu_2'(j) = \mu_2(j) Q_j^2 \mu(j)$ for all $j \in I_2' \setminus \{i\}$. This contradicts the fact that for all $k \in \widetilde{I}_2 \subseteq I_2' \setminus \{i\}$, $\mu(k)\ Q_k^2\ \mu_2(k)$. ■

Although many well-known mechanisms satisfy the conditions in Theorem **??**, the celebrated top trading cycles (TTC) is not one of them. Indeed, we may have a wasteful equilibrium if we use the TTC mechanism in the second round. We illustrate this point in the following example.[23]

Example 1 *Let* $S = \{s_1, s_2, s_3, s_4\}$, $S^1 = \{s_1\}$, $q = (1,1,1,1)$, $I = \{i_1, i_2, i_3, i_4\}$, *and* $h(i_1) = h(i_2) = h(i_3) = h(i_4) = \emptyset$. *Priorities and preferences are given as*

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$
i_1	i_2	i_3	i_3
i_2	i_4	i_2	i_2
i_3	i_3	i_1	i_1
i_4	i_1	i_4	i_4

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}
s_4	s_1	s_2	s_2
s_1	s_3	s_4	$\emptyset$
$\emptyset$	$\emptyset$	$\emptyset$	

.

Let $\Psi = (\varphi^1, \varphi^2)$ *be a system where* φ^1 *is a serially fair mechanism and* φ^2 *is TTC. Consider the following strategy profile:*

- *In the first round, students submit their true preferences over* $S^1 \cup \emptyset$: $s_1 P^1_{i_1} \emptyset$, $s_1 P^1_{i_2} \emptyset$, $\emptyset P^1_{i_3} s_1$ *and* $\emptyset P^1_{i_4} s_1$.
- *Students participating in the second round submit their true preferences over the available schools.*

At this preference profile, Ψ *will select the following matching:* $\mu = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & \emptyset \\ i_1 & i_3 & i_2 & \emptyset & i_4 \end{pmatrix}$. *One can verify that no student can get a better assignment by deviating. Indeed,* (P^1, P^2) *is a SPNE. At* μ, *the seat in* s_4 *is wasted.* ■

It is possible to further generalize the above observation. Using Example **??**, we can also show that for any given $\Psi = (\varphi^1, \varphi^2)$ where φ^1 is a fair mechanism and φ^2 is TTC, all the SPNE outcomes of the preference revelation game associated with Ψ are wasteful.

Theorem 6 *Let* $\Psi = (\varphi^1, \varphi^2)$ *be a system such that*

- φ^1 *is individually rational, non-wasteful, mutually fair, and favors higher ranks, or*
- φ^1 *is individually rational, non-wasteful, population monotonic, weakly non-bossy, and fair.*

[23] Once again, we do not describe this mechanism for brevity.

Every SPNE outcome of the preference revelation game associated with Ψ *leads to a matching* μ *in which there does not exist any* (i, j) *pair such that* $\mu(j) \in S^1$, $\mu(j)\ P_i\ \mu(i)$ *and* $i \succ_{\mu(j)} j$.

Proof. Let $Q = (Q_i^1, Q_i^2)_{i \in I}$ be an SPNE profile and μ be the associated equilibrium outcome. First note that $\mu(k) R_k h(k)$ for all $k \in I$. Otherwise, Q cannot be SPNE (Theorem **??**). Suppose there exist two agents $i, j \in I$ such that $\mu(j) \in S^1$, $\mu(j) P_i \mu(i)$ and $i \succ_{\mu(j)} j$. Let $\mu(j) = s$, $I' = \{i' \in I | s P_{i'} \mu(i')$ and $i' \succ_s j\}$ and $\widehat{i} \in I'$ be the student who has the highest priority for s among the ones in I'. We claim that submitting $Q' : s\ Q'\ \emptyset\ Q'\ x$ for all $x \in S^1 \diagdown \{s\}$ in round 1 is a profitable deviation for $\widehat{i}$.

Let $\varphi^1(Q^1) = \mu_1$, $\varphi^1(Q', Q^1_{-\widehat{i}}) = \widetilde{\mu}_1$ and $\widetilde{I}_1 = \{k \in I | \mu_1(k) \neq \widetilde{\mu}_1(k) = s\}$. Since φ^1 is individually rational then either $\widetilde{\mu}_1(\widehat{i}) = s$ or $\widetilde{\mu}_1(\widehat{i}) = \emptyset$. If $\widetilde{\mu}_1(\widehat{i}) = s$ then Q' is a profitable deviation for $\widehat{i}$. If $\widetilde{\mu}_1(\widehat{i}) = \emptyset$ then $|\widetilde{\mu}_1^{-1}(s)| = q_s$, $k \succ_s \widehat{i}$ for all $k \in \widetilde{\mu}_1^{-1}(s)$, $\widetilde{\mu}_1(j) \neq s$ and $\widetilde{I}_1 \neq \emptyset$. Otherwise mutual fairness[24] and/or non-wastefulness of φ^1 would be violated. Suppose $\widetilde{\mu}_1(\widehat{i}) = \emptyset$.

φ^1 *favors higher ranks and is mutually fair:* Since φ^1 favors higher ranks, $s\ Q_k^1\ x$ for all $x \in S^1 \cup \emptyset$ and $k \in \widetilde{I}_1$. Then, μ_1 cannot be mutually fair.

φ^1 *is fair, weakly non-bossy, and population monotonic:* Consider the profile $(Q'', Q^1_{-\widehat{i}})$ where $Q'' : \emptyset\ Q''\ x$ for all $x \in S^1$. By weak non-bossiness and individual rationality, $\varphi^1(Q'', Q^1_{-\widehat{i}}) = \widetilde{\mu}_1$. By population monotonicity and individual rationality, $\widetilde{\mu}_1(l) = s Q_l^1 \mu_1(l)$ for all $l \in \widetilde{I}_1$. Since $l \succ_s \widehat{i} \succ_s j$ and $s Q_l^1 \mu_1(l)$ for all $l \in \widetilde{I}_1$, fairness is violated in matching μ_1. ■

Theorems **??** and **??** can also be interpreted as giving the conditions for a mechanism to induce equilibrium outcomes that are fair, individually rational, and non-wasteful. Consequently, this also implies the result of Ergin and Sönmez (2006), who show that every equilibrium outcome of the who show that every equilibrium outcome of the preference revelation game induced by the Boston mechanism is fair, individually rational, and non-wasteful. It is easy to check that the Boston mechanism is indeed mutually fair, individually rational, non-wasteful, population monotonic, and weakly non-bossy.

Theorem 7 *Let* $\Psi = (\varphi^1, \varphi^2)$ *be a system such that* φ^1 *is individually rational, population monotonic, and*

- φ^2 *is individually rational, non-wasteful, monotonic, independent of irrelevant agents, weakly non-bossy, and fair, or*

- φ^2 *is individually rational, non-wasteful, mutually fair, and favors higher ranks.*

[24] Recall that mutual fairness is a weaker condition than fairness.

Every SPNE outcome of the preference revelation game associated with Ψ leads to a matching μ in which there does not exist any (i, j) pair such that $\mu(j) \in S^2$, $\mu(j)\ P_i\ \mu(i)$ and $i \succ_{\mu(j)} j$.

Proof. Let $Q = (Q_i^1, Q_i^2)_{i \in I}$ be an SPNE profile and μ be the associated equilibrium outcome. First note that $\mu(k)\ R_k\ h(k)$ for all $k \in I$. Otherwise, Q cannot be SPNE (Theorem **??**). Suppose there exist two agents $i, j \in I$ such that $\mu(j)\ P_i\ \mu(i)$, $\mu(j) \in S^2$ and $i \succ_s j$. Let $\mu(j) = s$. There are two cases: (1) i does not participate in the second round because either $\mu(i) \in S^1$ or $h(i) \neq \emptyset$; or (2) i participates the second round.

Suppose that i participates in the second round. Without loss of generality, let i be the agent with the highest priority for s among the ones who prefer s to his assignment, participating in the second round. Note that when no agent deviates from his strategy in the first round, the set of schools and agents in the second round do not change. Therefore, we can prove that i can benefit from deviating to $Q' : s\ Q'\ \emptyset\ Q'\ x$ for all $x \in S^2 \setminus \{s\}$ by following the same steps in the proof of Theorem **??**.

Now suppose that there does not exist an agent i' such that (1) $i' \in I^2$, (2) $s\ P_{i'}\ \mu(i')$ and (3) $i' \succ_s j$. Then $\mu(i) \in S^1$. We claim that submitting $(\widetilde{Q}, Q')$ is a profitable deviation for i where $\widetilde{Q} : \emptyset\ \widetilde{Q}\ x$ for all $x \in S^1$. Due to individual rationality $\varphi_i^1(\widetilde{Q}, Q') = \emptyset$. By population monotonicity and individual rationality, if $\varphi_j^1(\widetilde{Q}, Q_{-i}^1) \in S^1$ then $\varphi_j^1(Q^1) \in S^1$. Therefore, in the second round the set of agents is a subset of $I^2 \cup \{i\}$ and set of available seats weakly increases compared to the case in which i plays Q_i^1. Let I_2', $\overline{\mu}_2$ and $\widetilde{q}^2$ respectively be the set of agents, the selected matching, and the quota vector in round 2 when i submits $(\widetilde{Q}, Q')$. By individual rationality, $\overline{\mu}_2(i)$ is either s or $\emptyset$. Suppose $\overline{\mu}_2(i) = s$. Let $\widetilde{I}_2 = \{j \in I_2' | \mu(j) \neq \overline{\mu}_2(j) = s\}$. Since φ^2 is non-wasteful and mutually fair, $|\overline{\mu}_2^{-1}(s)| = q_s$, $\widetilde{I}_2 \neq \emptyset$, $k \succ_s i$ for all $k \in \overline{\mu}_2^{-1}(s)$, and $\overline{\mu}_2(i) \neq s$. Next, suppose $\overline{\mu}_2(i) = \emptyset$.

φ^2 favors higher ranks and is mutually fair: Since φ^2 favors higher ranks, $s\ Q_k^2\ x$ for all $x \in S^2 \cup \emptyset$ and $k \in \widetilde{I}_2$. Then, μ_2 cannot be mutually fair.

φ^2 is fair, weakly non-bossy, monotonic, and independent of irrelevant agents: By weak non-bossiness and individual rationality, $\overline{\mu}_2$ will be selected when i submits $\widetilde{P}_i^2 = \emptyset\ \widetilde{P}_i^2\ x$ for all $x \in S^2$. Let μ_2' be the outcome of φ^2 when we consider only agents in $I_2' \setminus \{i\}$, keeping everything else the same. By the independence of irrelevant agents, $\mu_2'(l) = \overline{\mu}_2(l)$ for all $l \in I_2' \setminus \{i\}$. By monotonicity, $\mu(l) Q_l^2 \mu_2'(l) = \overline{\mu}_2(l)$ cannot be true for any $l \in I_2' \setminus \{i\}$. This contradicts the fact that φ^2 is fair. ∎

5.1 Subgame Perfect Nash Equilibria of SD-DA in SCPwEXRS

In this subsection, we analyze the SPNE of the current sequential assignment system used in SCPwEXRS, where SD is applied in the first round and DA in the second round. SD is individually rational, non-wasteful, population monotonic, non-bossy, and strategy-proof. Moreover, it selects a fair (mutually fair) outcome when only the exam schools are available. DA is individually rational, non-wasteful, population monotonic, strategy-proof, and fair (mutually fair). Theorem **??** implies that every SPNE outcome of SD-DA leads to a non-wasteful and individually rational matching under true preferences.

Corollary 7 Every SPNE outcome of the preference revelation game associated with SD-DA leads to a non-wasteful and individually rational matching under agents' true preferences.

Proof. Follows from Theorem **??**. ■

In tSCPwEXRS, not all SPNE of the (preference revelation) game associated with SD-DA lead to a fair matching under agents' true preferences. However, every SPNE of this game leads to a matching where the priorities of the exam schools are respected under agents' true preferences.

Corollary 8 *Every SPNE outcome of the preference revelation game associated with SD-DA leads to a matching μ in which there does not exist any (i, j) pair such that $\mu(j) \in S^e$, $\mu(j) P_i \mu(i)$ and $i \succ_{\mu(j)} j$.*

Proof. Follows from Theorem **??**. ■

Recall that only those students who have not been assigned to an exam school participate in the second round. Since it is the last round and a strategy-proof mechanism is being used, agents cannot benefit from misreporting in this round. That is, it is a weakly dominant strategy for every students to submit his true preferences over the available schools in round 2. Without loss of generality, in the rest of this subsection, we assume that students act truthfully in the second round of SD-DA.

Let (S, I, P, P_S, q) be the associated college admission problem of the school choice problem with exam and regular schools, $(S, I, P, \succ, q, h)$, where for each school s, $i \ P_s \ j$ if and only if $i \succ_s j$. In particular, the main difference between the college admissions problem and the school choice problem is that in the college admissions problem, schools are active and have preferences over students, $P_S = (P_s)_{s \in S}$, whereas in the school choice problem, schools are passive and considered as objects to be consumed. A matching μ in a school choice problem is individually rational, non-wasteful, and fair if and only if it is stable for its associated college admissions problem (Balinski and Sönmez, 1999). Moreover, for each college admissions problem, there is a unique stable matching which is preferred to any other stable matching by every student. This matching is the well-known student-optimal stable matching. It follows, for the associated school choice problem that, the student-optimal stable matching is individually rational, non-wasteful, fair and preferred to any other such matching by every student. In the following proposition, we show that in any school choice problem with exam and regular schools, there exists at least one SPNE outcome of the game associated with SD-DA that is (weakly) preferred to any individually rational, non-wasteful and fair matching by all students.[25]

Proposition 4 *In any SCPwEXRS, there always exists at least one SPNE outcome of the preference revelation game associated with the SD-DA mechanism that (weakly) Pareto dominates any individually rational, non-wasteful, and fair matching.*

Proof. We show the existence of a SPNE outcome that (weakly) Pareto dominates the student-optimal stable matching.[26] Given a problem, denote the corresponding student-optimal stable matching with μ. Then consider the following strategy profile $\widetilde{P} = (\widetilde{P}_i^1, \widetilde{P}_i^2)_{i\in I}$:

- Student i submits $\widetilde{P}_i^1 : \mu(i)\ \widetilde{P}_i^1\ x\widetilde{P}_i^1\ \emptyset$ for all $x \in S^e \backslash \{\mu(i)\}$ in the first round if $\mu(i) \in S^e$,
- Student i submits $\widetilde{P}_i^1 : \emptyset\ \widetilde{P}_i^1\ x$ for all $x \in S^e$ in the first round if $\mu(i) \in S^r$, and
- Student i submits his true preferences over the regular schools and $\emptyset$ in the second round whenever he is active, i.e. $\widetilde{P}_i^2 = P_i|(S^r \cup \emptyset)$.

Denote the outcome of SD-DA mechanism under this preference profile by ν. We first show that $\nu(i)\ R_i\ \mu(i)$ for all $i \in I$. Under preference profile $\widetilde{P}$ only students in $\mu^{-1}(s)$ apply to each $s \in S^e$. Hence, $\nu(i) = \mu(i)$ for all $i \in \bigcup\limits_{s\in S^e} \mu^{-1}(s)$. Consider problem $(S^r, I^2, (q_s)_{s\in S^r}, (\widetilde{P}_i^2)_{i\in I^2}, \succ_{S^r})$ where $I^2 = \bigcup\limits_{s\in S^r\cup\emptyset} \mu^{-1}(s)$ and $\widetilde{P}_j^2 = P_j|(S^r \cup \emptyset)$. Define $\nu' : I^2 \to S^r \cup \emptyset$ and $\mu' : I^2 \to S^r \cup \emptyset$ such that $\nu'(i) = \nu(i)$ and $\mu'(i) = \mu(i)$ for all $i \in I^2$. One can verify that μ' and ν' are individually rational, fair and non-wasteful for the problem $(S^r, I^2, (q_s)_{s\in S^r}, (\widetilde{P}_i^2)_{i\in I^2}, \succ_{S^r})$. Moreover, ν' is the student-optimal individually rational, fair, and non-wasteful matching. Hence, $\nu(i)\ R_i\ \mu(i)$ for all $i \in I^2$. In Example **??** we show that there are problems where some students strictly prefer ν to μ.

[25]In Example 2, we provide a problem in which one of the SPNE may Pareto dominates any individually rational, non-wasteful and fair matching.

[26]Equivalently, this is the most-preferred individually rational, fair, and non-wasteful matching for each agent.

To show that $\widetilde{P}$ is SPNE, we first look at the subgames in the second round. Each subgame can be considered as an independent school choice problem. Since truthtelling is a weakly-dominant strategy under DA, submitting true preferences in the second round is a NE in each subgame.

Now we analyze the strategies in the first round. First consider $i \in I^2$. Since $\nu(i)\ R_i\ \mu(i)$, $\nu^{-1}(s) = \mu^{-1}(s)$ for all $s \in S^e$ and μ is fair and non-wasteful, all the seats of the exam schools that i prefers to $\nu(i)$ are filled by students with better exam scores in ν. Therefore, all exam schools that i prefers to $\nu(i)$ fill their seats before i's turn and i cannot be assigned to a better exam school no matter what he submits.

Next, consider a student j, who is assigned to an exam school. Since $\nu^{-1}(s) = \mu^{-1}(s)$ for all $s \in S^e$ and μ is fair and non-wasteful, all the seats of the exam schools that i prefers to $\nu(i)$ are filled by students with better exam score. We should also check whether he can be assigned to a preferred regular school. If j deviates and participates in the second round, then we should consider the subgame where $I^2 \cup j$ is active. Without loss of generality, we change the preference profile of j by placing $\emptyset$ just before $\mu(j)$ and represent it with P'_j. Let $\widetilde{P}'_j = P'_j|(S^r \cup \emptyset)$. It is easy to see that if j can be assigned to a better school than $\mu(j)$ in $(S^r, I^2 \cup j, (q_s)_{s\in S^r}, (\widetilde{P}^2_i)_{i\in I^2\cup j}, \succ_{S^r})$, then he will be assigned to the same school in $(S^r, I^2 \cup j, (q_s)_{s\in S^r}, (\widetilde{P}'_j, (\widetilde{P}^2_i)_{i\in I^2}), \succ_{S^r})$. Define a new matching $\mu'' : \bigcup_{s\in S^r\cup\emptyset} I^2 \to S^r \cup \emptyset$ such that $\mu''(i) = \mu(i)$ for all $i \in I^2 \backslash j$ and $\mu''(j) = \emptyset$. Then it is easy to see that μ'' is individually rational, fair, and non-wasteful in problem $(S^r, I^2 \cup j, (q_s)_{s\in S^r}, (\widetilde{P}'_j, (\widetilde{P}^2_i)_{i\in I^2}), \succ_{S^r})$. As a consequence of the rural hospital theorem (Roth 1986), in all the stable matchings the set of students assigned to a real school will be the same. Therefore, DA will not assign j to a better school than $\mu(j)$ if he deviates and participates in the second round. ■

We illustrate the result of Proposition **??** in the following example.

Example 2 *Let $S^e = \{s_1\}$, $S^r = \{s_2, s_3\}$, $q = (1, 1, 1)$ and $I = \{i_1, i_2, i_3\}$. Priorities and preferences are given as*

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
i_1	i_2	i_3
i_2	i_i	i_2
i_3	i_3	i_1

P_{i_1}	P_{i_2}	P_{i_3}
s_2	s_3	s_2
s_1	s_2	s_3
$\emptyset$	$\emptyset$	$\emptyset$
s_3	s_1	s_1

.

We can find the student-optimal individually rational, fair, and non-wasteful matching by applying DA *mechanism to the associated school choice problem. The rounds of the DA mechanism are as follows (the tentatively held students are indicated by an asterisk):*

	s_1	s_2	s_3
Round 1		i_1^*, i_3	i_2^*
Round 2		i_i^*	i_2, i_3^*
Round 3		i_1, i_2^*	i_3^*
Round 4	i_1^*	i_2^*	i_3^*

The outcome of DA is $\mu = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_1 & i_2 & i_3 \end{pmatrix}$. *Consider the following strategy profile:*

- *Students participating in the first round submit the following preferences:* $s_1 \widetilde{P}^1_{i_1} \emptyset$, $\emptyset \widetilde{P}^1_{i_2} s_1$, *and* $\emptyset \widetilde{P}^1_{i_3} s_1$.
- *Students participating in the second round submit their true preferences over regular schools.*

This strategy profile is indeed a SPNE and the induced outcome is: $\mu' = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_1 & i_2 & i_3 \end{pmatrix}$. *This mathcing Pareto dominates the student optimal individually rational, fair, and non-wasteful matching* $\mu = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_1 & i_3 & i_2 \end{pmatrix}$. ■

Proposition **??** and Example **??** imply that every SPNE outcome of the preference revelation game associated with SD-DA does not necessarily lead to a non-wasteful, individually rational, and fair matching under agents' true preferences. On the other hand, we can relate every non-wasteful, individually rational, and fair matching under agents' true preferences to a SPNE outcome of the game associated with SD-DA.

Theorem 8 *Every non-wasteful, individually rational, and fair matching under agents' true preferences can be supported as a SPNE outcome of the game associated with SD-DA.*

Proof. We omit the proof as it is easy to modify the proof of Theorem **??** to show this result. ■

5.2 Subgame Perfect Nash Equilibria of TRSD

Recall that in TAP, the SD mechanism is applied in both rounds. SD is individually rational, non-wasteful, population monotonic, non-bossy, and strategy-proof. Moreover, it selects a fair (mutually fair) outcome when the available schools are not owned by participating agents in rounds 1 and 2. It follows from Theorems **??**, **??**, and **??** that every SPNE outcome of TRSD leads to a non-wasteful, individually rational, and fair matching under the true preferences.

Corollary 9 Every SPNE outcome of the preference revelation game associated with TRSD leads to a non-wasteful, individually rational, and fair matching under agents' true preferences.

We illustrate this result in the following example.

Example 3 *Let $S = \{a, b, c\}$, $q = (1, 1, 1)$, $I^e = \{e\}$, and $I^n = \{t_1, t_2\}$. Teacher e is currently working in school a, and the other two schools are tenured positions. The ranking based on test scores is given by: $c(t_1) > c(t_2) > c(e)$. True preferences and utilities of the teachers are given as:*

e	t_1	t_2	U
c	a	a	3
b	b	b	2
a	c	c	1
$\emptyset$	$\emptyset$	$\emptyset$	0

Since a strategy-proof mechanism is used in the last round, teachers can benefit from a deviation only in the first round. In round 1, strategies are: $bc\emptyset$, $cb\emptyset$, $b\emptyset c$, $c\emptyset b$, $\emptyset bc$, and $\emptyset cb$, where we read $bc\emptyset$ as b is ranked over c and c is ranked over $\emptyset$. By individual rationality, $\emptyset bc$ and $\emptyset cb$ give the same outcome, and we represent both strategies by $\emptyset$. The payoff tables are given below. Here, t_2 is the matrix player, t_1 is the column player, and e is the row player.

$bc\emptyset$

	$bc\emptyset$	$cb\emptyset$	$b\emptyset c$	$c\emptyset b$	$\emptyset$
$bc\emptyset$	1,2,1	1,1,2	1,2,1	1,1,2	**3,3,2**
$cb\emptyset$	1,2,1	1,1,2	1,2,1	1,1,2	**3,3,2**
$b\emptyset c$	**1,2,1**	1,1,2	**1,2,1**	1,1,2	1,0,2
$c\emptyset b$	1,2,1	1,1,2	1,2,1	1,1,2	**3,3,2**
$\emptyset$	**1,2,1**	1,1,2	**1,2,1**	1,1,2	1,0,2

$cb\emptyset$

	$bc\emptyset$	$cb\emptyset$	$b\emptyset c$	$c\emptyset b$	$\emptyset$
$bc\emptyset$	1,2,1	1,1,2	1,2,1	1,1,2	2,3,1
cb	1,2,1	1,1,2	1,2,1	1,1,2	2,3,1
$b\emptyset c$	1,2,1	1,1,2	1,2,1	1,1,2	2,3,1
$c\emptyset b$	1,2,1	1,1,2	1,2,1	1,1,2	1,0,1
$\emptyset$	**1,2,1**	1,1,2	**1,2,1**	1,1,2	1,0,1

$b\emptyset c$

	$bc\emptyset$	$cb\emptyset$	$b\emptyset c$	c	$\emptyset$
$bc\emptyset$	*3,2,3*	*1,1,2*	*3,2,3*	*1,1,2*	***3,3,2***
$cb\emptyset$	*3,2,3*	*1,1,2*	*3,2,3*	*1,1,2*	***3,3,2***
$b\emptyset c$	*1,2,0*	*1,1,2*	*1,2,0*	*1,1,2*	*1,0,2*
$c\emptyset b$	*3,2,3*	*1,1,2*	*3,2,3*	*1,1,2*	***3,3,2***
$\emptyset$	*1,2,0*	*1,1,2*	*1,2,0*	*1,1,2*	*1,0,2*

$c\emptyset b$

	$bc\emptyset$	$cb\emptyset$	$b\emptyset c$	$c\emptyset b$	$\emptyset$
$bc\emptyset$	*1,2,1*	*2,1,3*	*1,2,1*	*2,1,3*	*2,3,1*
$cb\emptyset$	*1,2,1*	*2,1,3*	*1,2,1*	*2,1,3*	*2,3,1*
$b\emptyset c$	*1,2,1*	*2,1,3*	*1,2,1*	*2,1,3*	*2,3,1*
$c\emptyset b$	*1,2,1*	*1,1,0*	*1,2,1*	*1,1,0*	*1,0,1*
$\emptyset$	***1,2,1***	*1,1,0*	***1,2,1***	*1,1,0*	*1,0,1*

$\emptyset$

	$bc\emptyset$	$cb\emptyset$	$b\emptyset c$	$c\emptyset b$	$\emptyset$
$bc\emptyset$	*3,2,3*	*2,1,3*	*3,2,3*	*2,1,3*	*2,3,0*
$cb\emptyset$	*3,2,3*	*2,1,3*	*3,2,3*	*2,1,3*	*3,3,0*
$b\emptyset c$	*1,2,0*	*2,1,3*	*1,2,0*	*2,1,3*	*2,3,0*
$c\emptyset b$	*3,2,3*	*1,1,0*	*3,2,3*	*1,1,0*	*3,3,0*
$\emptyset$	*1,2,0*	*1,1,0*	*1,2,0*	*1,1,0*	*1,0,0*

The bold payoffs represent the NE outcomes. (1,2,1) corresponds to the payoff of the school-optimal fair, non-wasteful, and individually rational matching and (3,3,2) corresponds to the payoff of the teacher-optimal fair, non-wasteful, and individually rational matching. Moreover, teacher-optimal and school-optimal matchings are the only two matchings that are fair, non-wasteful, and individually rational. ■

In the following theorem, we show that in TAP every non-wasteful, individually rational and fair matching under agents' true preferences can be associated with an SPNE outcome of TRSD.

Theorem 9 *Every non-wasteful, individually rational, and fair matching under agents' true preferences is led by a SPNE outcome of the preference revelation game associated with TRSD.*

Proof. Let μ be a non-wasteful, individually rational, and fair matching under agents' true preferences. Then consider the following strategy profile $Q = (Q_i^1, Q_i^2)_{i \in I}$ where Q_i^t is the submitted preferences in round $t \in \{1, 2\}$ such that

- if $\mu(i) \in S^t$ then $\mu(i) Q_i^1 x Q_i^1 \emptyset$ for all $x \in S^t \setminus \{\mu(i)\}$,
- if $\mu(i) \in S^c$ and $h(i) = \emptyset$ then $\emptyset Q_i^1 x$ for all $x \in S^t$ and $\mu(i) Q_i^2 x Q_i^2 \emptyset$ for all $x \in S^c \setminus \{\mu(i)\}$,

- if $\mu(i) = h(i) \neq \emptyset$ then $\emptyset Q_i^1 x$ for all $x \in S^t$,

- if $\mu(i) = h(i) = \emptyset$ then $\emptyset Q_i^1 x$ for all $x \in S^t$ and $\emptyset Q_i^2 x$ for all $x \in S^t$.

The outcome of this strategy profile is μ. By individual rationality, if $\mu(i) \in S$ then i cannot be better off by submitting a preference profile which makes him unassigned.

Consider the second round. Agent i participates in the second round if $h(i) = \emptyset$ and $\emptyset Q_i^1 x$ for all $x \in S^t$. Suppose there exists a teacher, j, among the ones participating round 2 who can get $sP_j\mu(j)$ by deviating from his strategies in Q where $s \in S^c = S^2$. Since μ is non-wasteful $|\mu^{-1}(s)| = q_s$. Moreover, by fairness, any student in $\mu^{-1}(s)$ either has a higher test score or is an existing teacher in s. Therefore, all seats of s are filled before j's turn in round 2 and she cannot get that school no matter what she submits. Therefore, in any subgame in round 2 a Nash equilibrium is selected under preference profile Q^2.

Now consider the first round. Suppose there exists a contractual teacher j who can get $sP_j\mu(j)$ by deviating from his strategies in Q. First note that $s \notin S^c$, because the system does not allow an agent with ownership to participate in the second round. Then, $s \in S^t$. Since μ is non-wasteful and fair, all seats of s are filled before j's turn and she cannot get that school no matter what she submits. Now we show that a new graduate cannot be better off by deviating. A new graduate cannot increase the number of available seats when each contractual teacher $k \in I^e$ submits Q_k^1. That is, the number of available seats in each round cannot be affected by the deviation of a new graduate. By fairness and non-wastefulness, any school that a new graduate prefers to her assignment is filled with either teachers with higher test scores or with existing teachers. Therefore, no matter what a new graduate submits, she cannot get a better school than his assignment in μ. ■

6 A Simpler Alternative System: Simultaneous Assignment via DA

In Section 4, we show that the main reason behind the deficiencies observed in the current systems may simply be due to the fact that assignments are done sequentially. These impossibilities motivate us to advocate one-round assignment systems over sequential assignment whenever it is feasible to do so.

In the assignment systems discussed in this paper, one of the important concerns is assigning agents to schools without violating the predetermined priorities. Additionally, decreasing the level of gaming and thereby encouraging agents to report their true preferences over schools is another practical concern. Fortunately, an easy solution is available. By simply reducing the two rounds into a single round and applying the agent-proposing DA, one can readily address these concerns.

For any problem, DA selects a fair, non-wasteful, and individually rational outcome, i.e., it is stable. Moreover, DA is immune to preference manipulation and respects the improvements in test scores (priorities). A natural question is whether or not there is another alternative which satisfies all these desirable features. The following result based on Alcalde and Barbera (1994) and Balinski and Sönmez (1999) gives a negative answer to this question and makes the case for DA as a remedy to the deficiencies of the systems used in the two applications we discussed.[27]

Theorem 10 *DA is the unique mechanism that is*

- *fair, individually rational, non-wasteful, strategy-proof, or*
- *fair, individually rational, non-wasteful, and respects improvements in priorities.*

7 Conclusion

Although strategic and distributional objectives in standard (one-round) assignment problems and sequential assignment problems are quite similar, we have shown that the latter type of problems may be fundamentally different and more challenging than the former type. We have shown that under sequential systems, the most desirable properties are lost even though they may be satisfied roundwise. Most remarkably, sequential systems are strategically vulnerable (even if they are strategy-proof roundwise) and force participants to make hard judgment calls about how to rank the available options in each round. As a result, these systems may lead to inefficient and even wasteful assignments. This suggests that even though sequential systems may arguably be easier to implement in practice (e.g., in the context of school choice), this convenience may come at an important cost. The alternative use of one-round systems, such as the DA mechanism, may help avoid these costs when doing so is feasible.

The recent transition in Turkey to such a system[28] may also provide support for our conclusions. In our July 2012 meeting with the former Minister of Education, Ömer Dinçer, we explained our concerns about

[27]It is well-known that the outcome of DA is not necessarily Pareto efficient. However, it Pareto dominates any other stable matching for a given problem.

[28]In the recently adopted new system in Turkey, all teachers are assigned via a serial dictatorship. The practice of hiring contractual teachers has been discontinued in recent years.

the existing practice. In that meeting Mr. Dinçer stated that he and his planning team were well aware of the challenges posed by the existing system and hinted at potential reforms to follow. Later in that year, the Turkish ministry of education announced that they will discontinue hiring for the contractual positions and that all the contractual teachers will be converted to tenured teachers.

References

[1] Atila Abdulkadiroğlu and Tayfun Sönmez, "House Allocation with Existing Tenants", Journal of Economic Theory 88 (1999), pp. 233-260.

[2] Atila Abdulkadiroğlu and Tayfun Sönmez, "School Choice: A Mechanism Design Approach", American Economic Review 93 (2003), pp. 729-747.

[3] Atila Abdulkadiroğlu, Parag Pathak and Alvin Roth, "Strategy-proofness versus Efficiency in Matching with Indifferences: Redesigning the NYC High School Match", American Economic Review 99 (2009), pp1954-1978.

[4] Jose Alcalde and Salvador Barbera, "Top Dominance and the Possibility of Strategy-Proof Stable Solutions to Matching Problems", Economic Theory 4 (1994), pp. 417-435.

[5] Michel Balinski and Tayfun Sönmez, "A Tale of Two Mechanisms: Student Placement", Journal of Economic Theory 84 (1999), pp. 73-94.

[6] Sebastian Braun, Nadja Dwenger, Dorothea Kübler and Alexander Westkamp, "Implementing Quotas in University Admissions: An Experimental Analysis", working paper.

[7] Haluk Ergin and Tayfun Sönmez, "Games of School Choice under the Boston Mechanism", Journal of Public Economics 90 (2006), pp. 215-237.

[8] David Gale and Lloyd Shapley, "College Admissions and the Stability of Marriage", The American Mathematical Monthly 69 (1962), pp. 9-15.

[9] Pablo Guillen and Onur Kesten, "Matching markets with mixed ownership: the case for a real-life assignment mechanism", International Economic Review 53 (2012), pp. 1027-1046.

[10] Onur Kesten, "On Two Competing Mechanisms for Priority Based Allocation Problems", Journal of Economic Theory 127 (2006), pp. 155-171.

[11] Fuhito Kojima and M. Utku Ünver, "The Boston School Choice Mechanism: An Axiomatic Approach", Economic Theory, 2013.

[12] D. G. McVitie and L. B. Wilson, "The Stable Marriage Problem", Communications of the ACM 14 (1971), pp. 486-490.

[13] Thayer Morrill, "An Alternative Characterization of Top Trading Cycles", Economic Theory, 2012.

[14] Alvin Roth, "On the Allocation of Residents to Rural Hospitals: A General Property of Two-Sided Matching Markets," Econometrica, 54 (1986), pp. 425–427.

[15] Alexander Westkamp, "An Analysis of the German University Admissions System", Economic Theory, 2012.

A Appendix

A.1 Assignment systems in Boston and NYC

In Boston there are three exam schools[29] which enroll around 25% of the seventh grade students.[30] In a given year, sixth grade students take the centralized exam before December and apply to one of these schools in the following year. A ranking of students are then obtained based on a combination of the exam scores and GPAs from the previous year. The assignment of the students to the exam schools are determined via the serial dictatorship mechanism induced by this ranking. Admitted students receive their acceptance letters from the exam schools by mid-March[31], and the assignment for the regular schools are determined via DA.

In New York City there are nine exam schools.[32] The assignments to the exam and regular schools are also implemented sequentially although students submit their preferences over both types of schools at the

[29] These schools are Boston Latin Academy, Boston Latin School and the John D. O'Bryant School of Mathematics and Science.

[30] In 2012-2013 school year 836 of 3,795 seventh grade students have enrolled to exam schools.

[31] Sixth graders can also apply to be transferred to another regular school after mid-March.

[32] These schools are Bronx High School of Science, Brooklyn Latin School, Brooklyn Technical High School, High School for Math, Science and Engineering at City College, High School of American Studies at Lehman College, Queens High School for the Sciences at York College, Staten Island Technical High School and Stuyvesant High School.

same time. Every year between 25,000 and 30,000 student take the Specialized High School Admission Test (SHSAT) which is then used to determine the assignments to the exam schools which enroll only about 5,000 annually. Students who take this test submit two different rank-order-lists to the central authority. In the first one they rank-list only the exam schools whereas in the second one they rank-list only the regular schools which do not require any test score. The admission decisions for the specialized high schools are determined based on the scores on SHSAT, while the admissions for the regular schools follow the outcome of DA. Both decisions are concurrently determined. The central authority aims to make placements to the specialized high schools first. Therefore, initially only those students who have been admitted to both an exam and a regular school are informed, and they are asked to make a choice between the two schools they are admitted to. Subsequently, students who are not assigned in this round are considered and they are assigned to the regular schools once again via DA.

A.2 Turkish Assignment System

The number of teachers assigned to tenured and contractual positions in 2009 and 2010 is presented in Table 2. For instance in December 2009, 8,850 tenured positions were filled by applicants in the first round. 6,323 of these applicants were existing teachers working in contractual positions. These contractual positions which became available as a consequence of assignments of existing teachers to the tenured positions were filled in the same month.

Time of the Assignment	Type of the Positions	Number of Positions Filled
February 2009	Tenured	8,285
March 2009	Contractual	6,323
December 2009	Tenured	8,850
December 2009	Contractual	6,323
June 2010	Tenured	10,000
July 2010	Contractual	9,000
December 2010	Tenured	30,000
December 2010	Contractual	6,843

Table 2. Number of Teachers Assigned to Tenured and Contractual Positions (2009-2010)

A.3 Examples

In the following two examples, we illustrate how SD-DA and TRSD mechanisms fail to satisfy the desired properties.

Example 4 *Let $S = \{s_1, s_2, s_3, s_4\}$, $S^e = \{s_3, s_4\}$, $S^r = \{s_1, s_2\}$, $I = \{i_1, i_2, i_3, i_4\}$ and $h(i_1) = h(i_2) = h(i_3) = h(i_4) = \emptyset$. All schools have one available seat, $q_s = 1$ for all $s \in S$. Let true preferences and test scores be as follows:*

$$\begin{array}{lrcl} s_2 P_{i_1} s_3 P_{i_1} s_1 P_{i_1} s_4 P_{i_1} \emptyset & c(i_1) & = & 90 \\ s_1 P_{i_2} s_4 P_{i_2} s_2 P_{i_2} s_3 P_{i_2} \emptyset & c(i_2) & = & 88 \\ s_3 P_{i_3} s_1 P_{i_3} s_2 P_{i_3} s_4 P_{i_3} \emptyset & c(i_3) & = & 85 \\ s_4 P_{i_4} s_2 P_{i_4} s_1 P_{i_4} s_3 P_{i_4} \emptyset & c(i_4) & = & 70 \end{array}$$

The set of available schools in round 1 is $S^1 = \{s_3, s_4\}$. The outcome selected in round 1 when all the agents act truthfully (straightforwardly) is $\mu_1(i_1) = s_3$, $\mu_1(i_2) = s_4$, $\mu_1(i_3) = \emptyset$ and $\mu_1(i_4) = \emptyset$. In round 2 the set of the available schools and set of the applicants allowed to participate are: $S^2 = \{s_1, s_2\}$ and $I^2 = \{i_3, i_4\}$. The outcome selected in round 2 when all the agents act truthfully (straightforwardly) is $\mu_2(i_3) = s_1$ and $\mu_2(i_4) = s_2$. The final outcome of the SD-DA mechanism is $\mu = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ i_3 & i_4 & i_1 & i_2 \end{pmatrix}$.

SD-DA is not Pareto efficient: *There exists another matching $\mu' = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ i_2 & i_1 & i_3 & i_4 \end{pmatrix}$ that Pareto dominates the outcome of the SD-DA mechanism, μ. It is worth to mention that μ' is a fair matching. That is, the outcome of the current mechanism is Pareto dominated by a fair matching.*

SD-DA is not Strategy-proof: *If i_2 ranks s_4 below $\emptyset$ in his list in round 1 then the final outcome will be $\mu' = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ i_2 & i_4 & i_1 & i_3 \end{pmatrix}$ and i_2 will be strictly better-off.*

SD-DA does not respect improvements: *If we take $c'(i_2) = 75$ then the outcome will be $\mu' = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ i_2 & i_4 & i_1 & i_3 \end{pmatrix}$ and $\mu'(i_2) P_{i_2} \, \mu(i_2)$. That is, when i_2 gets higher score he is assigned to a less preferred school.*

SD-DA is not fair: *$\mu(i_4) P_{i_1} \mu(i_1)$ and i_1 has higher priority for $\mu(i_4) = s_2$.*

SD-DA is wasteful: *Consider the same example with only one agent,* $I = \{i_1\}$. *SD-DA mechanism assigns* i_1 *to* s_3. *But* i_1 *prefers* s_2 *to its match* s_3 *and* s_2 *has an empty under the outcome of SD-DA.*

Consider the same example with the following modification, $S^r = \{s_3, s_4\}$ *and* $S^e = \{s_1, s_2\}$. *All the other things are kept the same. Then it is easy to see that DA-SD mechanism suffers from the same deficiencies as the SD-DA mechanism.* ■

Example 5 *Consider Example* **??** *with the following modifications:* $h(i_1) = s_1$, $h(i_2) = s_2$, $h(i_3) = h(i_4) = \emptyset$. *Take the same test scores for students* i_1, i_3 *and* i_4. *Only change the test score of* i_2 *to* $c(i_2) = 80$. *Then TRSD mechanism selects the following matching:* $\mu = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ i_4 & i_2 & i_1 & i_3 \end{pmatrix}$.

TRSD is not Pareto efficient: *There exists another matching* $\mu' = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ i_3 & i_4 & i_1 & i_2 \end{pmatrix}$ *that Pareto dominates the outcome of the TRSD mechanism* μ. *It is worth to mention that* μ' *is a fair matching. That is, the current mechanism is Pareto dominated by a fair matching.*

TRSD is not Strategy-proof: *If* i_3 *ranks* s_4 *below* $\emptyset$ *in the submitted preferences in round 1 then the final outcome will be* $\mu' = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ i_3 & i_4 & i_1 & i_2 \end{pmatrix}$ *and* i_3 *will be strictly better-off. Moreover none of the agents will be hurt.*

TRSD does not respect improvements: *If we take* $c'(i_3) = 75$ *then the outcome will be* $\mu' = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ i_3 & i_4 & i_1 & i_2 \end{pmatrix}$ *and* $\mu'(i_3) P_{i_3}\, \mu(i_3)$. *That is, when* i_3 *gets higher score he is assigned to a less preferred school.*

TRSD is not fair: $\mu(i_4) P_{i_3} \mu(i_3)$ *and* i_3 *has higher priority than* i_4 *for* $\mu(i_4) = s_1$.

TRSD is wasteful: *Consider the same example with only two agents,* $I = \{i_1, i_3\}$ *and* $S = \{s_1, s_3, s_4\}$ *where* $h(i_1) = s_1$ *and* $h(i_3) = \emptyset$. *The preference of agents are*

$$\begin{array}{lrcl} s_3 P_{i_1} s_1 P_{i_1} s_4 & c(i_1) & = & 90 \\ s_3 P_{i_3} s_1 P_{i_3} s_4 & c(i_3) & = & 85 \end{array}$$

The matching selected by the TRSD is $\mu'' = \begin{pmatrix} s_1 & s_3 & s_4 \\ \emptyset & i_1 & i_3 \end{pmatrix}$. *But* i_3 *prefers* s_1 *to its match* s_4 *and* s_1 *has an empty seat under* μ''. ■