Dynamic Matching in Large Markets*†

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October 1, 2013
PRELIMINARY DRAFT

Abstract

Many matching markets have dynamic features; for example, the assignment of children to public daycare centers, teachers to public schools, high-level bureaucrats to different regions, and even the well-known school choice problem (once student mobility, or sibling priorities, are taken into account). In a dynamic market, the priorities of one side of the market might depend on previous allocations, which generates incentives for manipulations. In a finite dynamic school choice problem, for example, no mechanism exists that is both stable and strategy-proof. We show that under suitable restrictions on the schools' priorities, the deferred acceptance mechanism (which is stable under truth-telling) is not manipulable if the market is large. Formally, if the schools' priorities satisfy a condition which we call *IPA*, then the fraction of players with incentives to manipulate the deferred acceptance mechanism approaches zero as the number of participants increases. Conversely, under a priority structure that fails to satisfy this condition, such as the one currently in place in the Danish daycare system, the deferred acceptance mechanism remains manipulable even in large markets.

JEL classification: C78, D61, D78, I20.

KEYWORDS: Large market, dynamic school choice, deferred acceptance mechanism.

*We thank the financial support from the Center for Research in the Foundations of Electronic Markets.

†Tumennasan thanks the Economics Departments at Harvard University and Boston College for its hospitality and the Danish Council for Independent Research for providing the travel grant (#12-128681).

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1 Introduction

Game theory has been used with a great deal of success to redesign assignment markets. The well-known problem of allocating students to public schools illustrates well the usefulness of the field of market design in practical applications.[1]

One important objective in the school choice literature has been to propose mechanisms that implement a stable matching, which are said to be free of "justified envyness" from parents. From a practical point of view, stability is a very desirable property of a matching, and it is argued that mechanisms that produce stable matchings are less likely to be subject to unraveling, and more prone to succeed.[2] For this reason, much attention has been given to the deferred acceptance mechanism, which is strategy-proof and implements the student-optimal stable matching. The mechanism has since been adopted in the New York and Boston public school systems.

Apart from a few recent exceptions, the literature on matching, and in particular, the literature on school choice problem has focused on the static many-to-one matching problem. However, in reality, many markets have dynamic features. In a separate paper (Kennes et al. (2012)), we introduced the problem of allocating children to public daycare centers, which can be viewed as a dynamic version of the school choice problem. Children are assigned to daycare centers, but parents can choose to keep their child at home for longer and they can switch daycare centers over time. Moreover, at any point in time, a daycare might have overlapping cohorts of children. In that paper, we proved an impossibility result: there is no mechanism that is stable and strategy-proof. In particular, contrary to the static many-to-one matching problem, the deferred acceptance mechanism is manipulable by students. A few recent papers also worked on similar dynamic centralized matching problems and have also proved negative results concerning stability and strategy-proofness.[3] Given the importance of a stable matching in many-to-one matching markets, the question of how to implement such allocations in dynamic markets became a relevant question from a theoretical and practical point of view.

In this paper, we study the incentives for manipulation in the daycare assignment problem when the market is large. The deferred acceptance mechanism yields a stable matching

[1]See Roth (2002), for example, or Abdulkadiroğlu and Sönmez (2003) for a seminal paper on the school choice problem.

[2]See Roth (2002), and Kojima and Pathak (2009).

[3]For example, Pereyra (2013) studies the allocation of teachers to public schools, and Dur (2011) considers a dynamic school choice problem in which the incentives for placing siblings are taken into account.

in dynamic markets when all participants report truthfully, but the mechanism is manipulable.[4] Here, we identify conditions for the implementation of this mechanism as the number of participants increases, assuming time-separable preferences. We show that if each daycare center's priorities over children are history dependent only through previously enrolled children, then the deferred acceptance mechanism can be implemented in large markets. Specifically, we show that the fraction of children who find it profitable to misreport their preferences when all other participants are reporting truthfully approaches zero as the total number of participants increases. Conversely, if the priorities of the daycares depend on the matching of the previous period through schools other than the one that a child is currently enrolled in, then the system becomes manipulable and there is no "large market relief." It has been argued already that non-strategy-proof mechanisms might be implemented successfully, provided that the strategic issues are not severe (see Kojima and Pathak (2009) and Budish and Cantillon (2012), for example). What our result suggests is that the Deferred Acceptance mechanism might be successfully implemented in practice.

Denmark, which illustrates our daycare assignment problem, adopts a priority structure in which each child that has not been allocated to any school will have a higher priority then any previously allocated student in the subsequent period, except in the exact daycare where this allocated student is. This rule is denoted "child care guarantee." If the deferred acceptance mechanism is applied period-by-period in a market with a priority structure that follows the Danish priorities, the system will be manipulable even in large markets. However, if we were to drop the child care guarantee, then the incentives for manipulation would vanish as the market increases.

Many other important markets share the dynamic properties of our daycare problem. For example, there is considerable mobility of children enrolled in public schools (Schwartz et al. (2009)). Moreover, in some places, the priorities of schools over children may be history-dependent: in Boston, children enrolled in preschools have a higher priority over other children in that same school, which generates excess demand for preschools. The scope for manipulation generates dissatisfaction and frustration on parents, as illustrated by The Boston Globe.[5] We show that using the deferred acceptance for the Boston schools with the current priority structure implies that the system is manipulable even in the large.

The theory of dynamic market design is very recent. Ünver (2010) studies the kidney exchange problem considering a dynamic environment in which the pool of agents evolves over time. Kurino (2013), Pereyra (2013), Dur (2011) and Kennes et al. (2012) study the centralized matching when there is overlapping generations of agents. Bloch and Cantala

[4]Kennes et al. (2012).
[5]Ebbert (2011).

(2011) study a dynamic matching problem, but focus on the long-run properties of different assignment rules. Our paper is also related to the literature of large matching markets, for example, Kojima and Pathak (2009) and Azevedo and Leshno (2013).

The paper is organized as follows, in section 2, we provide the model and the main definitions. We also describe a version of the deferred acceptance mechanism, from Kennes et al. (2012). In section 3 we examine the main properties of the mechanism in small economies. Section 4 contains the results for an economy with a continuum of agents. In section 5 we prove our main convergence result.

2 Model

2.1 Setup

Time is discrete and $t = -1, 0, \cdots, \infty$. There is a finite number of infinitely lived schools (daycares in our example). Let $S = \{s_1, \cdots, s_m\}$ be the set of schools. Let $\bar{S} = S \cup \{h\}$ where h stands for the option of staying home. Let $r = (r^s)_{s \in \bar{S}}$ be the vector of capacities. We assume that $r^s < \infty$ for all $s \in S$ and $r^h = \infty$.

We will consider two different set-ups. Mainly, we are interested in environments in which the set of children is finite, but large. However, for expositional reasons, we will also consider the case in which the set of children is uncountable. In either one of the two set-ups, we assume that each child can attend school when she is one and two years old.[6] The type of a child i is a triplet $(t_i, \succ_i, x_i)$ where t_i is the period in which child i is one year old, $\succ_i$ is the child's strict preference ordering over pairs of schools, and $x_i = (x_i^s)_{s \in S} \equiv [0, 1)^m$ is the child's priority score vector at period t_i. This priority vector is fixed at the child is birth year, but may change as the child gets older if the schools' priorities are history-dependent, as we will often assume. We will specify how this priority score vector evolves over time as a function of the allocations. If a child i has a priority score vector x_i and a child j, born in the same period has a priority score vector given by x_j, with $x_i^s > x_j^s$, for some $s \in S$, we have that child i has a higher priority than child j at school s. The set of all possible types is $\mathcal{T}$.

At period t, a finite set of one year old children I_t arrives, i.e., $i \in I_t$ if and only if $t_i = t$. Consequently, at any period t the set of school-age children is $I_{t-1} \cup I_t$. As time passes the set of school-age children evolves in the "overlapping generations" (OLG) fashion. Let $I = (I_t)_{t=-1}^{\infty}$. An economy is a pair $E = (I, r)$.

Now let us define the matching in our setting.

[6] The restriction to two periods is for simplicity and will not affect our main results.

Definition 1 (Matching)**.** *A period t matching μ_t is a correspondence $\mu_t : I_{t-1} \cup I_t \cup \bar{S} \to I_{t-1} \cup I_t \cup \bar{S}$ such that*

1. *For all $i \in I_{t-1} \cup I_t$, $\mu_t(i) \in \bar{S}$*
2. *For all $s \in \bar{S}$, $|\mu_t(s)| \leq r^s$ and $\mu_t(s) \subset I_{t-1} \cup I_t \cup \emptyset$*
3. *For all $i \in I_{t-1} \cup I_t$, $i \in \mu_t(s)$ iff $s = \mu_t(i)$.*

A matching μ is a collection of period matchings: $\mu = (\mu_{-1}, \mu_0, \cdots, \mu_t, \cdots)$.

We use the notation $\mu(i)$ to denote the pair of schools that i is matched with under matching μ: $\mu(i) = \big(\mu_{t_i}(i), \mu_{t_i+1}(i)\big)$. Let M_t be the set of period t matchings.

For technical convenience we assume that at period -1 every child stays home, i.e., the schools start their operation at period 0. Consequently, all matchings we consider have a common period -1 matching in which all school age children are matched with h.

Children's Preferences

The notation (s, s') denotes the allocation in which a child is placed at school s at age 1 and at school s' at age 2. We write $(s, s') \succeq_i (\bar{s}, \bar{s}')$ if either $(s, s') \succ_i (\bar{s}, \bar{s}')$ or $(s, s') = (\bar{s}, \bar{s}')$. Throughout the paper, we maintain the following assumption on preferences:

Assumption 1 (Weak Separability)**.** *If $(s, s) \succ_i (s', s')$ for some i, s and s', then $(s, s'') \succ_i (s', s'')$ and $(s'', s) \succ_i (s'', s')$ for any $s'' \neq s'$.*
Let $\mathcal{WS}$ be the set of preferences satisfying Assumption 1.

This assumption means that there is no externality (complementarity) from attending different schools. Specifically, attending an inferior school and a different second school is always less attractive than attending a superior school and the same second school. Note that the assumption does not rule out the possibility of complementarity from attending the same school for two periods. In other words, our assumption allows for switching costs: a child with weakly separable preferences might prefer attending an inferior school for two periods rather than attending two different superior schools in different periods.

Now let us define a stronger version of the weak separability assumption which rules out the possibility of the case we have discussed above.

Definition 2 (Separability)**.** *If $(s, s) \succ_i (s', s')$ for some i, s and s', then $(s, s'') \succ_i (s', s'')$ and $(s'', s) \succ_i (s'', s')$ for any s''.*

Switching costs are consistent with separability, as long as they are not very large. We here remark that the sole purpose of the separability assumption is to simplify the presentation of some of our examples, i.e., none of our results rely on this stronger assumption.

Lastly, let us define the concept of isolated preferences $P_i(\mu^{t-1})$ which depends on the original preferences of the child and the previous period's matching. This concept is first defined by Kennes et al. (2012) and is particularly useful to find stable matchings.

Definition 3 (Isolated Preference Relation)**.** *For given μ_{t-1}, isolated preference relation $P_i(\mu_{t-1})$ is a binary relation satisfying*

1. *For $\forall i \in I_t : sP_i(\mu_{t-1})s'$ if and only if $(s, s) \succ_i (s', s')$ for any $s \neq s' \in \bar{S}$*
2. *For $\forall i \in I_{t-1} : sP_i(\mu_{t-1})s'$ if and only if $(\mu_{t-1}(i), s) \succ_i (\mu_{t-1}(i), s')$ for any $s \neq s' \in \bar{S}$.*

Let $P(\mu_{t-1}) = (P_i(\mu_{t-1}))_{i \in I_{t-1} \cup I_t}$.

Schools' Priorities and History Dependence

We have already discussed that x_i is the priority score vector of child i at period t_i.

Assumption 2 (Strict Priorities)**.** *For any two players i, j in $I_{t-1} \cup I_t$, $x_i^s \neq x_j^s$ for all $s \in S$.*

Given the dynamic nature of our problem, in our model we will consider the case in which the priority score of child i at period $t_i + 1$ depends on the previous period's matching. For instance, in the Danish system currently in use, the schools give the highest priorities to their currently enrolled children. Given the importance of this restriction on the Danish system, and on its natural appeal, i.e. children will not be forced out of a school, we will maintain this assumption throughout our paper. To incorporate this restriction in our model, we define the *priority score function* of child i at school s as $X_i^s : M_{t_i-1} \cup M_{t_i} \to [0,1]^n$ for all $i \in I$ and $s \in S$. For now we assume that the priority score of a child changes only at the school she has attended in the previous period.

Assumption 3 (Independence of Past Attendances (IPA))**.** *Each school's priority score function satisfies the following conditions:*

1. *$X_i^s(\mu_{t_i-1}) = x_i$ for all μ_{t_i-1}.*
2. *(Priority for currently enrolled children)*
$$X_i^s(\mu_{t_i}) = \begin{cases} 1+\delta & \text{if } i \in \mu_{t_i}(s) \\ x_i & \text{otherwise} \end{cases}.$$
where δ is an arbitrarily small positive number.

This assumption states that a child who is matched to school s when she is one will have the highest priority score at school s when she is two. In addition, the child's priority score at any other school remains the same unless she was matched to that school at the age of 1. Here, observe that the attendees of any school s at some period t will have the same priority score of 1 at the school in the following period. This assumption, as we will see later, does not cause any problem to run the version of the deferred acceptance algorithm used in this paper– note that given assumption 2 we will not have the problem of having a positive mass of students with the same score competing for limited vacancies in a given school.

In the current Danish assignment system *IPA* is not satisfied: the schools give priority to two year old children who have not attended any school in the previous period over one year old children as well as over the two year old children who have attended school in the previous period. Although we assume the *IPA* condition for the majority of this paper, we will examine the current Danish priority system closely when we study the incentives to manipulate the deferred acceptance mechanism. For this reason we define the current Danish priority system.

Definition 4 (Danish Priorities)**.** *A priority scoring system is* Danish *if the priority score function for each school s and each student i satisfies the following 2 conditions :*

1. $X_i^s(\mu_{t_i-1}) = x_i^s$ *for all* μ_{t_i-1}.
2. $X_i^s(\mu_{t_i}) = \begin{cases} 1+\delta & \text{if } i \in \mu_{t_i}(s) \\ 1 & \text{if } i \in \mu_{t_i}(h) \\ x_i & \text{otherwise} \end{cases}$
where, recall, δ is an arbitrarily small positive number.

In the Danish priority scoring system a child who stays at home when she is young will have a priority score of 1 in all schools in the following period. Consequently, by staying home at age 1, a child jumps ahead of almost all children (except the school's previous period's attendees) in the priority ranking of any school at age 2. Here, we impose the following tie-breaking rule. If the mass of children applying to a particular school is greater than the school's capacity (something that could potentially happen if many students stay home when they are 1 year old), the priorities of schools over these children will follow the original priority score vector of these children.

Periodwise Deferred Acceptance Mechanism

Kennes et al. (2012) adapts the Gale and Shapley deferred acceptance mechanism to the daycare assignment problem when the schools priorities are strict. This mechanism which we call the *periodwise deferred acceptance* (PDA) runs starting period 0 as period -1 matching is fixed. In period 0 the school age children report their *isolated* preferences based on the previous period's matching which is fixed. Now the period 0 matching is found by running the following algorithm in finite rounds. Each child submits the complete list with her isolated preferences for that particular period t, i.e. taking as given the matching in period $t-1$.

Round 1: Each child is tentatively assigned to her most preferred school according to her isolated preferences. Each school tentatively assigns its spots to the proposers according to its priority ranking. If the number of proposers to school s is greater than the number of available spots r_s, then the remaining proposers are rejected.

In general, at:
Round k: Each child whose first option was rejected in the previous round is tentatively assigned to her next choice according to her isolated preferences. Each school considers the pool of children who it had been holding plus the current proposers. Then it tentatively assigns its spots to this pool of children according to its priority ranking. The remaining proposers are rejected.

The algorithm terminates when no proposal is rejected and each child is assigned her final tentative assignment.

In period 1, the schools' priorities scores are updated based on the period 0 PDA matching. In addition, all the school age children in this period report their isolated preferences based on the period 0 PDA matching. Now using the algorithm described above, we find the period 1 PDA.

In each period $t \geq 2$ we can run the above algorithm recursively based on the preceding period's PDA matching.

The PDA mechanism yields a unique matching in each economy. We will use the notation μ^{DA} for the PDA matching.

Here we remark that to run the PDA mechanism at any given period one only uses the information up till that period. This is very important property to have if one uses the PDA mechanism in practice. In addition, Kennes et al. (2012) show that the PDA mechanism always yields a strongly stable matching — the stability concept adapted to the dynamic setting of the daycare assignment.

Let the period t threshold score of school s corresponding to the PDA matching be p_t^s such that

$$p_t^s = \begin{cases} 0 & \text{if } \left|\mu_t^{DA}(s)\right| < r^s \\ \inf_{i \in \mu_t^{DA}(s)} X_i^s(\mu_{t-1}^{DA}) & \text{otherwise} \end{cases}$$

We use the following notations: $p_t = (p_t^s)_{s \in \bar{S}}$ and $p = (p_t)_{t=1}^{\infty}$.

3 Truth Telling in Small Economies

It is well known that in static settings, the student proposing DA mechanism is strategy-proof. Kennes et al. (2012)[7] show that the PDA mechanism is not strategy-proof. In fact, in some small markets a child finds it profitable to misrepresent her preferences even when everyone else reports her preferences truthfully.

We use the following notations: μ^{DA} is the PDA matching if every player reports truthfully; $\hat{\mu}^{DA}$ for the resulting PDA matching if player i misreports her preferences alone.

Theorem 1 (TT is not optimal when others report truthfully)**.** *In some small markets some children have incentives to misreport her preferences even when everyone else reports her preferences truthfully.*

Proof. Consider the following example: there are 3 schools $\{s, s_1, s_2\}$ and each school have a capacity of one child. There is no school-age child until period $t-1$. Suppose $I_{t-1} = \{i\}$, $I_t = \{i_1, i_2\}$, $I_{t+1} = \{i'\}$ and $I_\tau = \emptyset$ for all $\tau \geq t+2$. In addition, suppose that

$$\begin{array}{ccccccc} x_i^s & > & x_{i'}^s & > & x_{i_1}^s & > & x_{i_2}^s \\ x_i^{s_1} & > & x_{i_1}^{s_1} & > & x_{i_2}^{s_1} & > & x_{i'}^{s_1} \\ x_i^{s_2} & > & x_{i_1}^{s_2} & > & x_{i'}^{s_2} & > & x_{i_2}^{s_2} \end{array}$$

We consider two preference profiles which differ from each other in child i_1's preferences. Each child's preferences are *separable*. Child i's top choice is (s, s). The preferences of children i_2 and i' satisfy the following conditions:

$$\begin{array}{ccccc} (s_2, s_2) & \succ_{i_2} & (s_1, s_1) & \succ_{i_2} & (s, s) \\ (s_2, s_2) & \succ_{i'} & (s, s) & \succ_{i'} & (s_1, s_1) \end{array}$$

Child i_1's preference ordering is $\succ_{i_1}^1$ under preference profile 1 and is $\succ_{i_1}^2$ under profile 2. These preferences are given as follows:

$$\begin{array}{ccccc} (s, s) & \succ_{i_1}^1 & (s_1, s_1) & \succ_{i_1}^1 & (s_2, s_2) \\ (s, s) & \succ_{i_1}^2 & (s_2, s_2) & \succ_{i_1}^2 & (s_1, s_1) \end{array}$$

In addition, suppose $(s_2, s) \succ_{i_1}^1 (s_1, s_1)$.

Under profile 1, the PDA matching is as follows: $\mu^{t-1}(i) = \mu^t(i) = s$, $\mu^t(i_1) = \mu^{t+1}(i_1) = s_1$, $\mu^t(i_2) = \mu^{t+1}(i_2) = s_2$, $\mu^{t+1}(i') = s$ and $\mu^{t+2}(i') = s_2$.

[7] Their result is even stronger: no strongly stable and strategy proof mechanism exists.

Under profile 2, the PDA matching $\hat{\mu}$ is as follows: $\hat{\mu}^{t-1}(i) = \hat{\mu}^t(i) = s$, $\hat{\mu}^t(i_1) = s_2$, $\hat{\mu}^t(i_2) = s_1$, $\hat{\mu}^{t+1}(i_1) = s$, $\hat{\mu}^{t+1}(i_2) = s_1$, $\hat{\mu}^{t+1}(i') = s_2$ and $\hat{\mu}^{t+2}(i') = s_2$.

Under profile 1, child i_1's matching is (s_1, s_1) if she reports her preference truthfully but it is (s_2, s) if she misreports her preference as if under profile 2. But $(s_2, s) \succ^1_{i_1} (s_1, s_1)$. Consequently, child i_1 misreports her preferences under profile 1. Hence, the PDA mechanism does not induce truth telling. □

Lemma 1. *No child born in period -1 can profitably manipulate the PDA mechanism. In addition, if a child i born in period $t \geq 0$ can successfully manipulate the PDA mechanism then the following must be true:*

$$(\hat{\mu}^{DA}_{t_i+1}(i), \hat{\mu}^{DA}_{t_i+1}(i)) \underbrace{\succ_i}_{1} (\mu^{DA}_{t_i+1}(i), \mu^{DA}_{t_i+1}(i)) \underbrace{\succeq_i}_{2} (\mu^{DA}_{t_i}(i), \mu^{DA}_{t_i}(i)) \underbrace{\succeq_i}_{3} (\hat{\mu}^{DA}_{t_i}(i), \hat{\mu}^{DA}_{t_i}(i)) \quad (1)$$

and

$$(\mu^{DA}_{t_i}(i), \hat{\mu}^{DA}_{t_i+1}(i)) \underbrace{\succ_i}_{4} (\mu^{DA}_{t_i}(i), \mu^{DA}_{t_i+1}(i)).$$

Proof. See Appendix C. □

This lemma shows that to manipulate the PDA mechanism successfully one will have to accept a worse first period school in order to improve her second period allocation. This is indeed true even in the Danish priority system in which *IPA* is not satisfied. However, there is a very important difference in terms of the information required for manipulation. But before discussing this let us present an example of a system with the Danish priority structure in which truth telling is not an equilibrium.

Example 1 (TT not an equilibrium)**.** *Consider the following example: there are 2 schools $\{s_1, s_2\}$ and each school has a capacity of one child. Suppose $I_{-1} = \{i_{-1}\}$, $I_0 = \{i_0\}$, $I_1 = \{i_1\}$ and $I_\tau = \emptyset$ for all $\tau \geq 2$. All children's top choice is s_1, but worst choice is h. Each child's preferences are separable but satisfies the following condition*

$$(h, s_1) \succ (s_2, s_2).$$

In addition suppose that

$$x^s_{i_{-1}} > x^s_{i_1} > x^s_{i_0}$$

for each $s = s_1, s_2$. Here we assume that the priority system is Danish.

If everyone reports truthfully, child i_0's allocation is (s_2, s_2) under the PDA mechanism. Instead, suppose that i_0 reports that her first choice is s_1, but second choice is h. In this *case, child i_0 obtains (h, s_1) under the PDA mechanism. Hence, child i_0 has an incentive to manipulate.*

The two examples, one in Theorem 1 and Example 1, reveal the differences of successful manipulation in different priority systems in terms of required sophistication. In the Danish system by staying home when a child is young, she jumps ahead of almost everyone in all schools' priorities. This manipulation is relatively simple as it only involves one action which is staying home when young which ultimately improves the child's priority score relative to others' scores. On the other hand, manipulating the PDA mechanism in systems satisfying *IPA* is rather difficult. To see this let us concentrate in the example presented in Theorem 1. When child i_1 misreports her preferences, the child (in our case i') who were matched to school s in period $t+1$ under truth telling will still have priority over i_1 at school s. In other words child i_1's priority score at s does not improve at all no matter what she does. This means child i_1's manipulation must benefit child i' so that she never applies to s. This of course is possible in the example we considered, but the child must be rather sophisticated to see through all the possible effects of her manipulation.

4 Daycare Assignment with a Continuum of Agents

In this part we assume that the set of children born in period t, $\bar{I}_t = t \times \mathcal{WS} \times [0,1)^n$, is a continuum mass of students. We assume that $\bar{I}_t$ is common for each $t \geq -1$. Let $\bar{\nu}$ be a (probability) measure on $\bar{I}_t$ where $t = -1, \cdots, \infty$. Notice that we are assuming that the set of players born in each period is identical and the distribution of the new born children is identical in each period.[8] Let $\bar{r}$ be the vector of capacities. A continuum economy is $\bar{F} = (\bar{\nu}, \bar{r})$.

Assumption 4 (Strict Priorities). *For any $s \in S$, the measure of the children who has the same priority at this school is 0, i.e., $\bar{\nu}(\{i : t_i = t \,\&\, x_i^s = e\}) = 0$ for any $t \geq -1$ and $e \in [0,1)$.*

Assumption 5 (Market Thickness). *The probability measure $\bar{\nu}$ has a full support, i.e., for any $t \geq -1$, preference profile $\succ \in \mathcal{WS}$ and any $x \ll x' \ll \mathbf{1}$,*

$$\bar{\nu}(\{i : t_i = t \,\&\, \succ_i = \succ \,\&\, x \leq x_i \leq x'\}) > 0.$$

The assumption above means that the market is thick in the sense that the type space is sufficiently rich.

[8]Relaxing this assumption does not affect the main results of the paper but the notations will be considerably more complicated.

Definition 5 (Matching)**.** *A period t matching $\bar{\mu}_t$ is a function $\bar{\mu}^t : \bar{I}_t \cup \bar{I}_{t-1} \cup \bar{S} \to \bar{I}_t \cup \bar{I}_{t-1} \cup \bar{S}$ such that*

1. *For all $i \in \bar{I}_{t-1} \cup \bar{I}_t$, $\bar{\mu}_t(i) \in \bar{S}$*
2. *For all $s \in \bar{S}$, $\bar{\nu}\left(\bar{I}_{t-1} \cap \bar{\mu}_t(s)\right) + \bar{\nu}\left(\bar{I}_t \cap \bar{\mu}_t(s)\right) \leq \bar{r}^s$ and $\bar{\mu}_t(s) \subset \bar{I}_{t-1} \cup \bar{I}_t \cup \emptyset$*
3. *For all $i \in \bar{I}_{t-1} \cup \bar{I}_t$, $i \in \bar{\mu}_t(s)$ iff $s = \bar{\mu}_t(i)$.*
4. *The period t matching is right continuous, i.e., for any sequence of children $i^k = (\tau, \succ, x^k)$ where $\tau = t-1, t$ converging to $i = (\tau, \succ, x)$, we can find some large K such that $\bar{\mu}_t(i^k) = \bar{\mu}_t(i)$ for all $k > K$.*

A matching $\bar{\mu}$ is a collection of period matchings: $\bar{\mu} = (\bar{\mu}_{-1}, \bar{\mu}_0, \cdots, \bar{\mu}_t, \cdots)$.

As in the small economy case, we assume that in period -1 everyone stays home. We use the following notations: $\bar{\nu}\left(\bar{\mu}_t(s)\right) \equiv \bar{\nu}\left(\bar{I}_{t-1} \cap \bar{\mu}_t(s)\right) + \bar{\nu}\left(\bar{I}_t \cap \bar{\mu}_t(s)\right)$ and $\bar{\mu}(i) \equiv \left(\bar{\mu}_{t_i}(i), \bar{\mu}_{t_i+1}(i)\right)$.▌

Periodwise Deferred Acceptance Mechanism

The *periodwise deferred acceptance* (PDA) runs starting period 0 as period -1 matching is fixed. In period 0 the school age children report their *isolated* preferences based on the previous period's matching which is fixed. Now the period 0 matching is found by the following algorithm in maybe infinite rounds.

Round 1: Each child proposes to her first choice according to her isolated preferences. Each school tentatively assigns its spots to the proposers according to its priority ranking. Specifically, if the measure of proposers to school s is greater than its capacity $\bar{r}_s$, then it rejects the proposers who has a priority score $X_i^s(\bar{\mu}_{-1}^{DA})$ strictly below the minimum threshold $\bar{p}_0^{1s}$ at which the measure of the proposers above this threshold is equal to the school's capacity $\bar{r}^s$. Let $\bar{p}_0^1 = (\bar{p}_t^{1s})_{s \in S}$.

In general, at:
Round k: Each child who was rejected in the previous round proposes to her next choice according to her isolated preferences. Each school considers the pool of children who it had been holding plus the current proposers. Then it tentatively assigns its spots to this pool of children according to its priority ranking. Specifically, if the measure of proposers to school s is greater than its capacity $\bar{r}_s$, then it rejects the proposers who has a priority score $X_i^s(\bar{\mu}_{-1}^{DA})$ strictly below the minimum threshold $\bar{p}_0^{ks}$ at which the measure of the proposers above this threshold is equal to the school's capacity $\bar{r}^s$. Let $\bar{p}_0^k = (\bar{p}_0^{ks})_{s \in S}$.

The algorithm terminates when no proposal is rejected and each child is assigned her final tentative assignment. Let $\bar{p}_0 = (\lim_{k\to\infty} \bar{p}_0^{ks})_{s\in S}$.

In period 1, the schools' priorities scores are updated based on the period 0 PDA matching. In addition, all the school age children in this period report their isolated preferences based on the period 0 PDA matching. Now using the algorithm described above, we find the period 1 PDA. Let $\bar{p}_1$ be the threshold vector corresponding to period 1 PDA matching.

In each period $t \geq 2$ we can run the above algorithm recursively based on the preceding period's PDA matching. Let $\bar{p}_t$ be the threshold vector corresponding to period t PDA matching. Also let $\bar{p} = (\bar{p}_t)_{t=0}^{\infty}$.

The PDA mechanism yields a unique matching in each economy. We will use the notation $\bar{\mu}^{DA}$ for the PDA matching.

4.1 Truth Telling in Continuum Economies

Now we consider the continuum economies and show that when the PDA mechanism is used no player has an incentive to misrepresent her preferences as long as the others report their preferences truthfully.

Theorem 2 (TT Equilibrium)**.** *For any continuum economy, no player has an incentive to misreport her preferences in the PDA mechanism when every other child reports her preferences truthfully.*

Proof. Suppose that player i can successfully manipulate the PDA mechanism in some continuum economy $\bar{F}$. Let player i misreport her preferences as $\succ_i' \neq \succ_i$. Let the economy which results from i's misreporting be $\check{F}$. Let $\bar{\mu}^{DA}$ and $\check{\mu}^{DA}$ be the PDA matchings in $\bar{F}$ and $\check{F}$, respectively. Let the threshold scores corresponding to the $\bar{\mu}^{DA}$ and $\check{\mu}^{DA}$ be $\bar{p}$ and $\check{p}$. Since the two economies differ in only player i's preferences and given that the measure of each player is 0, we have that $\bar{p} = \check{p}$.

In a similar way to Lemma 1, we obtain that

$$(\bar{\mu}_{t_i}^{DA}(i), \check{\mu}_{t_i+1}^{DA}(i)) \succ_i (\bar{\mu}_{t_i}^{DA}(i), \bar{\mu}_{t_i+1}^{DA}(i)).$$

The above two relation means that $\check{p}_{t_i+1}^{s} \leq x_i^s < \bar{p}_{t_i+1}^{s}$ which is a contradiction. □

In the theorem above, we assumed that the priorities of the schools satisfy the *IPA* condition. Now we consider the Danish priority scoring system.

Theorem 3 (Manipulation under Danish Scoring System). *If the priority scoring system is Danish, then the PDA mechanism is manipulable even in continuum economies.*

Proof. Consider an economy in which the threshold score at some school s corresponding to the PDA matching is $\bar{p}$ with $\bar{p}^s_t > 0$, $\bar{p}^s_{t+1} > 0$ and $t \geq 0$. Consider a preference type $\succ_o \in \mathcal{WS}$ under which (s, s) is the the most preferred bundle, and $(h, s) \succ_o (s', s'')$ for all $s' \neq s$ and $s'' \neq s$. Now consider any child i with $\succ_i = \succ_o$, $t_i = t$ and $x_i < \min\{\bar{p}^s_t, \bar{p}^s_{t+1}\}$. Clearly, child i does not attend s by reporting her preferences truthfully. However, if she reports s as her first choice and h as her second choice, then she will stay home when she is one but attends s when she is 2. This means that child i has a profitable manipulation. In addition, thanks to Assumption 5, the measure of the children who has a profitable manipulation is strictly positive. □

Example 2 (Boston Pre-School System). *To be included...*

5 Large Markets and Convergence

Consider a finite economy $E = (I, r)$. We now define the measure for each finite economy E based on its empirical distribution. Specifically, the measure of each player i is $\tilde{\nu}(\{i\}) = 1/|I_{-1}|$. On the other hand, let the capacities of the schools be $\tilde{r} = r/|I_{-1}|$. Let $\tilde{F} = (\tilde{\nu}, \tilde{r})$ be the economy corresponding to the finite economy E.

Definition 6. *A sequence of finite economies E^k converges to a continuum economy $\bar{F}$ if the sequence of economies $\tilde{F}^k = (\tilde{\nu}^k, \tilde{r}^k)$ corresponding to E^k satisfies the following two conditions:*

1. *$\tilde{\nu}^k$ converges to $\bar{\nu}$ in weak* topology*

2. *$\tilde{r}^k$ converges to $\bar{r}$ in supremum norm.*

Here observe that if E^k converges to $\bar{F}$, then the ratio of the size of children born in any period t to the size of the children born in $t-1$ converges to 1.

Let $\bar{\mu}^{DA}$ and μ^{DA} be the PDA matchings of a continuum economy F and of a finite economy E. The period t distance between $\bar{\mu}^{DA}$ and μ^{DA} are as follows:

$$d_t(\mu^{DA}, \bar{\mu}^{DA}) = \parallel p_t - \bar{p}_t \parallel_\infty .$$

Let $d(\mu^{DA}, \bar{\mu}^{DA}) = \big(d_t(\mu^{DA}, \bar{\mu}^{DA})\big)_{t=-1}^{\infty}$.

Definition 7. *A sequence of PDA matchings μ^{DAk} converges to $\bar{\mu}$ if*

$$\lim_{k\to\infty} d_t(\mu^{DAk}, \bar{\mu}^{DA}) = 0 \text{ for all } t \geq 0.$$

Proposition 1. *If a sequence of finite economies E^k converge to a continuum economy $\bar{F}$, then the sequence of PDA matchings μ^k converges to $\bar{\mu}^{DA}$.*

Proof. See Appendix C. □

Let L_t be the set of the children born in period t who benefits by manipulating the PDA mechanism while the others report their preferences truthfully.

Theorem 4. *If a sequence of finite economies E^k converges to a continuum economy $\bar{F}$, then $\frac{|L_t^k|}{|I_t^k|} \to 0$ for each $t \geq 0$.*

Proof. See Appendix C. □

As a last remark, we note that a version of theorem 3 also holds for large, finite, markets. That is, if the priority scoring system is Danish, then the PDA is manipulable even as the market becomes large.

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Appendix A: Static Stability in Small Economies

To prove Proposition 1 we need some new definitions and results which we include in Appendices A and B. The proof of Proposition 1 is in Appendix C.

Fix a finite economy $E = (I, r)$ and a period $t-1$ matching μ_{t-1} of this economy. Now let us construct a new period t finite economy $\tilde{E}_t(\mu_{t-1})$ based on our original economy and μ_{t-1}. In this new economy the set of children is $\tilde{I}_t = I_t \cup I_{t-1}$ and each child i is defined by a pair $(P_i(\mu_{t-1}), X_i(\mu_{t-1}))$. Let $\mathcal{Q}$ be the all possible rankings of $\bar{S}$.

In this new economy, observe that for any two players i and j their priorities cannot satisfy the following conditions: $X_i^s(\mu_{t-1}) = X_j^s(\mu_{t-1}) < 1$ for any s.

Definition 8 (Static Stability)**.** *Period t matching μ_t is statically stable in economy $\tilde{E}(\mu_{t-1})$ if there exists no school-child pair (s, i) such that*

1. $sP_i(\mu_{t-1})\mu_t(i)$,
2. $|\mu_t(s)| < r^s$ *or/and* $X_i^s(\mu_{t-1}) > X_j^s(\mu_{t-1})$ *for some* $j \in \mu_t(s)$

From Gale and Shapley (1962) each period $t \geq 0$ PDA matching μ_t^{DA} is statically stable in economy $\tilde{E}(\mu_{t-1}^{DA})$.

Appendix B: Static Stability in Continuum Economies

Let us fix an economy $(\bar{\nu}, \bar{r})$. Fix any period $t \geq 0$ and a period $t-1$ matching $\bar{\mu}_{t-1}$. Now let us construct a new period t continuum economy $\hat{F}(\mu_{t-1}) = (\hat{\nu}, \hat{r} = \bar{r})$ based on our original economy and $\bar{\mu}_{t-1}$. In this new economy the set of children is $\hat{I}_t = \bar{I}_t \cup \bar{I}_{t-1}$ and each child i is defined by a pair $(P_i(\bar{\mu}_{t-1}), X_i(\bar{\mu}_{t-1}))$. Let $\mathcal{Q}$ be the all possible rankings of $\bar{S}$. In addition, for each Q let $I_{t-1}(Q, \bar{\mu}_{t-1}) = \{i \in I_{t-1} : P_i(\bar{\mu}_{t-1}) = Q\}$. Similarly, define $I_t(Q, \bar{\mu}^{DA}_{t-1})$. Now $\hat{I}_t$ is distributed on $\mathcal{Q} \times [0,1]^n$ according to a measure $\hat{\nu}$ where

$$\hat{\nu}\left(\{i \in \hat{I}_t : P_i(\bar{\mu}_{t-1}) = Q \,\&\, x \leq X_i(x_i, \bar{\mu}_{t-1}) \leq x'\}\right) = \bar{\nu}\left(\{i \in I_{t-1}(Q, \bar{\mu}_{t-1}) : x \leq X_i(x_i, \bar{\mu}_{t-1}) \leq x'\}\right) + \bar{\nu}\left(\{i \in I_t(Q, \bar{\mu}_{t-1}) : x \leq X_i(x_i, \bar{\mu}_{t-1}) \leq x'\}\right)$$

for all $x, x' \in [0,1]^n$ where $x \ll x'$.

Now observe that $\hat{\nu}$ has a full support because the second term in the equation above is always positive for all $x, x' \in [0,1]^n$ where $x \ll x'$. In addition, $\nu(\{i \in \hat{I}_t : X^s_i(\bar{\mu}_{t-1}) = x\}) = 0$ for all $x < 1$ and $s \in S$.

Definition 9. *Period t matching $\bar{\mu}_t$ is statically stable in economy $\hat{F}(\bar{\mu}_{t-1})$ if there exists no school-child pair (s,i) such that*

1. $sP_i(\bar{\mu}_{t-1})\bar{\mu}_t(i)$,
2. $\hat{\nu}(\mu_t(s)) < \bar{r}^s$ *or/and* $X^s_i(\bar{\mu}_{t-1}) > X^s_j(\bar{\mu}_{t-1})$ *for some* $j \in \bar{\mu}_t(s)$

Lemma 2. *For any economy $\hat{F}(\mu_{t-1})$, there exists a unique statically stable matching.*

Proof. We already pointed out that $\hat{\nu}$ has a full support and $\nu(\{i \in \hat{I}_t : X^s_i(\bar{\mu}_{t-1}) = x\}) = 0$ for all $x < 1$ and $s \in S$. Therefore, all the requirements for Theorem 1 of **?** is satisfied, hence $\hat{F}$ has a unique statically stable matching.

□

Lemma 3. *For any economy $\hat{F}(\mu^{DA}_{t-1})$, $\bar{\mu}^{DA}_t$ is a unique statically stable matching.*

Proof. This is a direct consequence of Lemma 3 and Proposition A1 of **?**. □

Appendix C: Proofs

Proof of Proposition 1. Take any sequence of DA matchings μ^{DAk} and the corresponding sequence of threshold scores p^k. For this proof we will use an induction argument. Assume that for all $\tau = 0, \cdots, t-1$, $p^k_\tau \to_{k\to\infty} \bar{p}_\tau$. Now we show $p^k_t \to_{k\to\infty} \bar{p}_t$.

At period t, consider $\hat{F}(\bar{\mu}^{DA}_{t-1})$ and $\tilde{E}^k(\mu_{t-1})$. Now based on $\tilde{E}^k(\mu_{t-1})$ let us define economy $\tilde{F}(\mu_{t-1}) = (\tilde{\nu}, \tilde{r})$ where the measure $\tilde{\nu}$ is a measure satisfying $\tilde{\nu}(\{i\}) = 1/|I_{-1}|$, and $\tilde{r} = r/|I_{-1}|$. Because $\mu^k_{t-1} \to_{k\to\infty} \mu^{DA}_{t-1}$, any sequence p^k_{t-1} converges to $\bar{p}_{t-1}$. Consequently, the sequence of measures $\tilde{\nu}^k_t$ must converge to $\hat{\nu}_t$ in the weak* sense. Then Theorems 2(ii) and 2(iii) of **?** yield that $p^k_t \to \bar{p}_t$. This completes the proof. □

Proof of Lemma 1. First let us show that any child born in period −1 cannot manipulate the PDA mechanism profitably. To see this, recall that these children's matching in period −1 is exogenously determined and to determine the period 0 matchings, the PDA mechanism uses the isolated preferences. In addition, because the DA mechanism is strategy proof in static settings, by misreporting no child born in period -1 improves in terms of her isolated preferences.

Relation 3 in 1 follows directly from the fact that the PDA mechanism is strategy-proof in terms of isolated preferences. Now observe that $\mu^{DA}(i) \succeq_i (\mu^{DA}_{t_i}(i), \mu^{DA}_{t_i}(i))$ because child i has the highest priority at school $\mu^{DA}_{t_i}(i)$ in period $t_i + 1$. This and Assumption 1 yields Relation 2 in 1. Now we show relation 4. First observe that Relation 3 and $\mu^{DA}(i) \succeq_i (\mu^{DA}_{t_i}(i), \mu^{DA}_{t_i}(i))$ yields that $\mu^{DA}(i) \succeq_i (\hat{\mu}^{DA}_{t_i}(i), \hat{\mu}^{DA}_{t_i}(i))$. This relation implies that $\hat{\mu}^{DA}_{t_i}(i) \neq \hat{\mu}^{DA}_{t_i+1}(i)$ because i successfully manipulates the PDA mechanism. Also $\mu^{DA}_{t_i}(i) \neq \hat{\mu}^{DA}_{t_i+1}(i)$. Otherwise, Assumption 1 and Relation 3 imply that $(\mu^{DA}_{t_i}(i), \mu^{DA}_{t_i}(i)) \succeq_i \hat{\mu}^{DA}(i)$ which contradicts with i successfully manipulating the PDA mechanism. Furthermore, $\mu^{DA}_{t_i+1}(i) \neq \hat{\mu}^{DA}_{t_i+1}(i)$. Otherwise, Assumption 1 and Relation 3 imply that $\mu^{DA}(i) \succeq_i \hat{\mu}^{DA}(i)$ which contradicts with i successfully manipulating the PDA mechanism. Now Assumption 1, $\hat{\mu}^{DA}_{t_i}(i) \neq \hat{\mu}^{DA}_{t_i+1}(i)$ and Relation 3 imply that

$$(\mu^{DA}_{t_i}(i), \hat{\mu}^{DA}_{t_i+1}(i)) \succ_i (\hat{\mu}^{DA}_{t_i}(i), \hat{\mu}^{DA}_{t_i+1}(i)).$$

This proves relation 4 because $\hat{\mu}^{DA}(i) \succ_i \mu^{DA}(i)$.

Now we show relation 1. On contrary assume $(\mu^{DA}_{t_i+1}(i), \mu^{DA}_{t_i+1}(i)) \succ_i (\hat{\mu}^{DA}_{t_i+1}(i), \hat{\mu}^{DA}_{t_i+1}(i))$. Then this, $\mu^{DA}_{t_i}(i) \neq \hat{\mu}^{DA}_{t_i+1}(i)$, and Assumption 1 give

$$(\mu^{DA}_{t_i}(i), \mu^{DA}_{t_i+1}(i)) \succ_i (\mu^{DA}_{t_i}(i), \hat{\mu}^{DA}_{t_i+1}(i)).$$

Combining the last two relations we reach a contradiction with $\hat{\mu}^{DA}(i) \succ_i \mu^{DA}(i)$. Hence, relation 1 is proved.

□

Proof of Theorem 4. Suppose that player i in finite economy E can manipulate the PDA mechanism. Let player i's matchings under truthful reporting and manipulating be $\mu^{DA}(i)$

and $\hat{\mu}^{DA}(i)$, respectively. From Lemma 1 we know that

$$(\mu_{t_i}^{DA}(i), \hat{\mu}_{t_i+1}^{DA}(i)) \succ_i (\mu_{t_i}^{DA}(i), \mu_{t_i+1}^{DA}(i)).$$

For notational simplicity let $\hat{\mu}_{t_i+1}^{DA}(i) = s$. Also let p and $\hat{p}$ be the threshold scores corresponding to $P(\mu_{t_i-1}^{DA})$ and $P(\hat{\mu}_{t_i-1}^{DA})$, respectively. Consequently,

$$\hat{p}_{t_i+1}^{s} \leq x_i^s < p_{t_i+1}^{s}.$$

In other words, if a child i can manipulate the PDA mechanism then there must exist a school $s \in S$ such that the inequality above is satisfied. Therefore, to prove the theorem it suffices to show that at each $t \geq 1$, s and $\epsilon > 0$, there exists high enough $\bar{k}$ such that for all $k \geq \bar{k}$, there exists no child with $t_i = t-1$, $|x_i^s - \bar{p}_t^s| \geq \epsilon$ and $\hat{p}_{t_i+1}^{ks} \leq x_i^s < p_{t_i+1}^{ks}$.

Suppose that the statement above is false. This means that for some $t \geq 1$, s, $\epsilon > 0$ and any $\bar{k}$, there exists $k \geq \bar{k}$ and i with $t_i = t-1$, $|x_i^s - \bar{p}_t^s| \geq \epsilon$ and $\hat{p}_t^{ks} \leq x_i^s < p_t^{ks}$. In other words, we can choose a subsequence of economies E^{k_j}, such that in each economy in this sequence, there exists player i^{k_j} who is born in period $t-1$, $|x_{i^{k_j}}^s - \bar{p}_t^s| \geq \epsilon$ and $\hat{p}_t^{k_j s} \leq x_{i^{k_j}}^s < p_t^{k_j s}$. Clearly, E^{k_j} converges to $\bar{F}$ in weak* sense. This means that $p_t^{k_j s}$ must converge to $\bar{p}_t^s$. Now consider the sequence of finite economies $\hat{E}^{k_j}$ which differs from E^{k_j} only in that player i^{k_j}'s preference type is the one she reports in her manipulation. Because in each of these economies only one player's preference type is changed, $\hat{E}^{k_j}$ converges to $\bar{F}$ in weak* sense. This means that $\hat{p}_t^{k_j s}$ must converge to $\bar{p}_t^s$. Recall that we already showed that $p_t^{k_j s}$ converges to $\bar{p}_t^s$. This means that as k_j increases, $x_{i^{k_j}}^s$ must be arbitrarily close to $\bar{p}_t^s$ because $\hat{p}_t^{k_j s} \leq x_{i^{k_j}}^s < p_t^{k_j s}$. Therefore, for a high enough k_j it cannot be $|x_{i^{k_j}}^s - \bar{p}_t^s| \geq \epsilon$ which is a contradiction.

This completes the proof as $\bar{\nu}\left(\{i : t_i = t \,\&\, x_i^s = e\}\right) = 0$ for any t and $e \in [0, 1)$.

□