Incomplete Information and Costly Signaling in College Admissions

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Abstract

We analyze a problem of college admissions with incomplete information about student skills. Colleges with observable qualities and students with private information are matched according to a decentralized mechanism where students can signal their abilities with costly observable signals. We characterize a separating symmetric equilibrium of this game where the number of potential matches is maximized and the best students enroll at the best colleges. Our closed form characterization allows us to conduct meaningful comparative statics analyses. We show that the effect of a change in the underlying parameters of the model is not symmetric across students. For instance, an increase in the number of students leads low skilled students to decrease the investment in signaling while high skilled students may increase it. Similar patterns arise when we analyze a change in the number of school places or college qualities. Finally, we analyze the gains of the signaling process by comparing equilibrium payoffs between this separating equilibrium and a pooling equilibrium with no signaling. Among other results, we show that under certain distributions of skills, a large enough demand for school places leads all colleges to get positive gains.

Keywords: College admissions, decentralized matching, incomplete information, coordination problems, costly signaling.
JEL Classification: D, C72, C78.

1 Introduction

A decentralized college admissions process is associated with coordination problems. This not only means that some agents may end up unmatched, but also that the mechanism may be ineffective to assign the best students to the best colleges. Not only the grade coordination among agents could explain these inefficiencies but also the presence of incomplete information. College qualities seem to be observable for all agents, however students' abilities are rather private information. This implies that colleges with observable qualities may face big indifferences among applicants depending on available information about student abilities. These indifferences may lead colleges to accept applicants already accepted by better colleges and remain unmatched depending on the choice of students.

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The literature on this issue shows that some simple matching mechanisms alleviate the effects of coordination problems in complete information environments.[1] These mechanisms exhaust the possibility of matching agents in a stable fashion and under certain conditions they also maximize the number of potential matches of the problem. Some of these matching mechanisms assure the stability of equilibrium assignments by restricting students to send only one application.[2] When agents can send multiple applications, unstable assignments may end up in equilibrium. However, it is easy to restore the stability of equilibrium matches by introducing a small application cost (Triossi, 2009). If the application cost is negligible, leading students to submit multiple applications, some dynamic mechanisms result effective to get stable equilibrium assignments.[3] As you can see, in environments with complete information it is relatively easy to guarantee the stability of equilibrium assignments and alleviate the problem of coordination of college admissions.

In contrast, in incomplete information environments we require additional conditions to alleviate the problem of coordination. In this setting, the role of signaling seems to be crucial to understand how colleges and students match each other in college admissions. For instance, Coles, et. al. (2010) introduce a cost-free signaling mechanism in decentralized matching problems with incomplete information about agents' preferences. Among other desirable properties, in equilibrium this mechanism increases the expected number of matches and the welfare of agents who signal their preferences. However, a cost-free signaling mechanism is not very realistic in decentralized college admissions. For instance, most selective colleges and universities in the US require a set of signals for college admissions. The test scores of either the SAT or ACT,[4] essay questions, recommendation letters and personal interviews are the most important requirements. A costly signaling model seems to be a good approach to analyze this problem, since a student has to spend significant amounts of effort (and money) in order get better signals and improve his chance of enrolling at college.

In this paper, we analyze a matching problem where students want to enroll at colleges with observable qualities. Student abilities are private information, however all agent know the prior distribution of student skills. In this setting, students want to enroll at the best universities while colleges want to enroll high skilled students. Agents are matched according to a simple decentralized matching mechanism called **Costly Signaling Mechanism** (CSM) which runs in two stages. In he signaling stage, students choose a costly observable score to signal their abilities. In the matching stage, colleges and students are matched according to a simple two-stage matching process. First, colleges simultaneously make one offer to a student; and after that, students collect their offers and simultaneously choose one offer among the available ones. The CSM induces an extensive form game that is characterized by an equilibrium assignment and a signaling strategy.

In order to understand the effects of the presence of incomplete information in college admissions, we analyze the benchmark matching problem with no signaling. In this setting, all students are ex-ante identical, since colleges only know the prior distribution of student

[1]These mechanisms are simple in the sense that they consist in only two stages. In the first stage, agents on one side of the market send (simultaneously) a proposal to the agents on the opposite side of the market. In the second stage, agents collect their offers and accept or reject (simultaneously) of of the available proposals.

[2]See Alcalde, Perez-Castrillo and Romero-Medina (1998) and Alcalde and Romero-Medina (2000).

[3]See Sotomayor, 2003; Romero-Medina and Triossi, 2010; and Haeringer, G. and Wooders, M., 2011.

[4]The SAT (Scholastic Assessment Test) and the ACT (American College Testing) are the most important standardized tests for college admissions in the USA.

abilities. Under these conditions, we characterize a symmetric equilibrium of this game whose agents' payoffs depend on the number of students, the expected value of student skills and colleges' qualities. This equilibrium has several interesting implications. First of all, in equilibrium colleges expect to enroll average students, since the matching process does not provide any additional information about student abilities. Secondly, we find that the probability of enrolling a student is decreasing in college qualities. Then only the highest quality college fills its school seat with probability one while the rest of agents may end up unmatched with positive probability. Finally, we show that an increase in the number of students increases colleges' payoffs while students' payoffs decrease.

Our main results regard with the analysis of a separating symmetric equilibrium of the game induced by the CSM where all students play according to the same signaling strategy. To sustain this separating equilibria, we consider a set of beliefs by which higher scored students are associated with higher skills. Under these beliefs, colleges form an interim ordinal preference relation on the set of students by which they prefer to enroll higher scored students. This implies that for each profile of student scores, there is unique equilibrium assignment in the matching stage of the CSM that is consistent with these beliefs. This equilibrium assignment is assortative, i.e. the highest scored student is matched with the best college; the second highest scored student is matched with the second best college; and so on.

In the signaling stage of the CSM, students take as given the assortative assignment of the matching stage and play a signaling game where they choose a costly observable score to signal their abilities. We characterize a symmetric pure strategy Nash equilibrium of this game. This equilibrium is characterized by an strictly increasing and continuous differentiable signaling strategy that depends on student skills. In equilibrium, no pair of students choose the same score then this symmetric signaling equilibrium induces a unique equilibrium matching that is assortative with respect to the true student skills. Hence in this separating equilibrium, the highest skilled student is matched with the best college; the second highest skilled student is matched with the second best college; and so on. Further, in this equilibrium the CSM it is maximized the number of potential matches and agents are induced to match efficiently in the sense that the best students enroll at the best colleges.

Our closed form characterization allows to conduct meaningful comparative static analyses on this separating equilibrium. Our main result shows that the effect of a change in the underlying parameters of the model is not symmetric across students, since they depend on student abilities. The first comparative statics exercise regards with the effect of a change in the number of students. An increase in the number of competitors (students) affects the probability of enrolling at college. This effect is not symmetric, since a low skilled student has a decrease in this probability while a high skilled student may has an increase. We show that an increase in the number of students leads low skilled students to decrease the investment in signaling while the high skilled students may increase it. To understand this result consider the following intuitive argument, a low skilled student should beat $N - k$ competitors to get a place at college c_k. When the number of competitors increases to $N' > N$, this student not only should beat $N' - k$ competitors to get the same school place but also has a big probability of facing new high skilled competitors. On the other hand, it is clear that a high skilled student should also beat more competitors to enroll at college but at the same time he has a small probability of facing new high skilled competitors. Hence, these two opposite effects may lead high skilled students to increase their probability of enrolling at college.

This result explains an interesting empirical fact in real-world college admissions. In the US is observed a decline in the mean SAT scores as the participation rate increases.[5] According to the College Board,[6] the mean SAT scores declines because more students of varied academic backgrounds are represented in the pool of test-takers. It is clear that this interpretation only considers the positive correlation between the SAT and students abilities to argue that an increase in the number of applicants systematically decreases the proportion of good test-takers in population. Our results suggest an alternative explanation based on the underlying signaling game of the problem. According to our model, an increase in the number of competitors not only leads low skilled students to decrease the investment in signaling but also increases the proportion of people who decide to do it. Then an increase in the number of competitors will eventually lead students to reduce the average investment in signaling with no change in the distribution of student abilities.

We also analyze the effect of a change in the number of school places and a change in college qualities on this equilibrium signaling strategy. These two experiment have very similar implications, and as in the previous case, the effect of these experiments is not symmetric across students. In particular, we show that an increase in the number of school places (in college qualities) leads low skilled students to increase the investment in signaling while the high skilled students may decrease it.

Finally, we analyze the gains of the CSM which are defined in a natural way as the difference in equilibrium payoffs between the separating signaling equilibria of the CSM and a symmetric equilibria of the college admissions problem with no signaling. We show that students' gains are strictly increasing with respect to student skills. However, this property of students gains does not guarantee students to avoid potential losses. Further, it is possible to show that under certain distributions of skills all students can have negative gains.

On the other hand, colleges' gains depend on expected values of order statistics. In general, this is a difficult issue, since for most distributions there are no closed form solutions for moments of order statistics. Thus, we analyze the particular case of exponentially distributed skills that allows us to calculate a closed form solution for colleges' gains. Even when the exponential model is a very particular case, it has very interesting implications. First of all, colleges' gains are monotone increasing in college qualities, i.e. the best college has the greatest gains; the second best college has the second greatest gains; and so on. Second, colleges' gains are monotone increasing in the number of students, i.e. all colleges benefit from an increasing demand for school places. Finally, we show that a sufficiently large demand for school places leads all colleges to get positive gains.

The rest of the paper is organized as follows; in Section 2, we describe the basic model and definitions; in Section 3, we analyze the benchmark college admissions problem with no signaling; in Section 4, we introduce CSM and equilibrium characterization; in Section 5, we conduct our exercises of comparative statics; in Section 6, we analyze the gains of the CSM; we present some conclusions in Section 7. Finally, all proofs are in the Appendix.

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[5] California Postsecundary Education Commission (CPEC), "SAT Scores and Participation Rate" at http://www.cpec.ca.gov/StudentData/50StateSATScores.asp.

[6] "43% of 2011 College-Bound Senior Met SAT College and Career Readness Benchmark" at http://press.collegeboard.org/releases/2011/43-percent-2011-college-bound-seniors-met-sat-college-and-career-readiness-benchmark

2 The model

There are $M \geq 1$ colleges and N students such that $M \leq N$. Let $C = \{c_1, c_2, ..., c_M\}$ denote the set of colleges with typical agent $c \in C$ and let $S = \{s_1, s_2, ..., s_N\}$ denote the set of students with typical agent $s \in S$. Each college $c \in C$ is characterized by an observable parameter $v_c > 0$, which is interpreted as the quality of the college c. With some abuse of notation, we use v_j to denote the quality of the college c_j. In order to simplify, we usually say "the student i" instead of "the student s_i" and "the college j" instead of "the college c_j". We assume without lost of generality (w.l.g) that colleges' qualities satisfy the following condition, $v_1 \geq v_2 \geq ... \geq v_M$.

Each student $s \in S$ is characterized by a parameter $\alpha_s \in [0, w]$ that denotes his skills or academic abilities. We say that a student s is more skilled than a student s' whenever $\alpha_s > \alpha_{s'}$. Students' skills are private information, this implies that only the student $s \in S$ knows the realization of his own parameter α_s. We assume that student skills are independently and identically distributed on some interval $[0, w]$ according to a strictly increasing and continuous differentiable cumulative distribution function F such that $F(0) = 0$ and $F(w) = 1$.[7] The distribution F has a continuous density $f = F'$ that satisfies $f(\alpha) > 0$ for all $\alpha \in (0, w)$. All elements of the model are common knowledge, i.e. the distribution of skills F; the number of students and colleges; and colleges' qualities.

2.1 The matching problem

For simplicity, we focus on the simplest one-to-one matching problem,[8] i.e. each college has only one available school seat. In this setting, an assignment is a matching between colleges and students which is a mapping that specifies a partner for each agent, allowing the possibility that some agents remain unmatched. Formally

Definition 1 *A matching μ is a mapping from the set $S \cup C$ onto itself such that:*

1. *If $\mu(s) \neq s$ then $\mu(s) \in C$;*
2. *If $\mu(c) \neq c$ then $\mu(c) \in S$; and*
3. *$\mu(s) = c$ if and only if $s = \mu(c)$.*

According to this definition, a student (college) with no partner is matched with himself (itself). In order to simplify, each student (college) get an utility equal to the quality (skills) of the partner. Let $U_s(\mu)$ and $U_c(\mu)$ be the utilities of the student s and the college c, respectively, under the matching μ. Then each student $s \in S$ has a payoff,

$$U_s(\mu) = \begin{cases} v_c \text{ if } \mu(s) = c. \\ 0 \text{ Otherwise} \end{cases} \tag{1}$$

Each college $c \in C$ has a payoff,

[7] All results hold when students' parameters are independently and identically distributed on the interval $[0, \infty)$ according to a strictly increasing and continuous differentiable cumulative distribution function F such that $F(0) = 0$ and $\lim F(w) = 1$ as $w \to \infty$.

[8] The model can be easily extended to many-to-one matching problems under the assumption that colleges form responsive preferences (Roth and Sotomayor, 1991).

$$U_c(\mu) = \begin{cases} \alpha_s \text{ if } \mu(c) = s. \\ 0 \ \text{ Otherwise} \end{cases} \tag{2}$$

We normalize the utility of the prospect of remaining unmatched to zero for colleges and students.

In two-sided matching literature, a college admissions problem is described by a three-tuple $(S, C, \succ)$, where S is a set of students, C is a set of colleges and $\succ = (\succ_{s_1}, ..., \succ_{s_N}; \succ_{c_1}, ..., \succ_{c_M})$ denotes a profile of ordinal preferences. In this problem, each agent $a \in S \cup C$ has a complete, strict and transitive preference relation $\succ_a$ over the set of agents on the opposite side of the market and the prospect of remaining unmatched.

It is easy to see that each student $s \in S$ has a preference relation $\succ_s$ over the set of colleges and the prospect of remaining unmatched $C \cup \{s\}$, such that: a) $c \succ_s s$ if and only if $v_c > 0$ and b) for all $c, c' \in C$, it is satisfied that $c \succ_s c'$ if and only if $v_c > v_{c'}$. Since college qualities are observable, all students have identical ordinal preferences. In a similar way, each college $c \in C$ has a preference relation $\succ_c$ on the set of students and the prospect of having a position unfilled $S \cup \{c\}$, such that: a) $s \succ_c c$ if and only if $\alpha_s > 0$ and b) for all $s, s' \in S$, it is satisfied that $s \succ_c s'$ if and only if $\alpha_s > \alpha_{s'}$. Let $\succeq_a$ denote the weak preference relation associated with $\succ_a$ for each agent $a \in S \cup C$. Thus, for any $c, c' \in C$, $c \succeq_s c'$ implies either $c \succ_s c'$ or $v_c = v_{c'}$. In a similar way, for any $s, s' \in S$, $s \succeq_c s'$ implies either $s \succ_c s'$ or $\alpha_c = \alpha_{c'}$.

A matching μ is **individually rational** whenever $\mu(a) \succeq_a a$ for all $a \in S \cup C$. A student-college pair (s, c) such that $\mu(s) \neq c$ **blocks** the matching μ if, $s \succ_c \mu(c)$ and $c \succ_s \mu(s)$. A matching μ is **stable** if it is individually rational and not blocked by any student-college pair. Let $\mathcal{E}(S, C, \succ)$ denote the set of **stable matchings** of the college admissions problem $(S, C, \succ)$.

3 The benchmark problem: College admissions with no signaling

We analyze the benchmark problem of college admissions with no signaling and incomplete information about student skills. In this setting, all students are ex-ante identical, since colleges only know the prior distribution of student skills. So the expected value of student abilities $E[\alpha]$ is the best estimation of student skills.

We consider that colleges and students are matched according to the following simple decentralized matching mechanism in two stages.

1. **Offers:** Each college $c \in C$ sends one message $m(c) \in S \cup \{c\}$. If $m(c) = s$, then the college c is making an offer to the student s. If $m(c) = c$, the college c is making no offer. Let $O(s) = \{c \in C : m(c) = s\} \cup \{s\}$ be the set of offers to the student s (note that a student always receives an offer from himself) ;

2. **Hiring:** Each student $s \in S$ chooses one of his available offers in $O(s)$.

Colleges and students play the game induced by this simple mechanism. In complete information environments with strict preferences, Alcalde and Romero-Medina (2000) show that this mechanism implements in **Subgame Perfect Equilibrium (SPE)** the set of stable matchings of college admissions problems. Thus this class of decentralized matching mechanisms exhausts the possibility of matching colleges and students in a stable way.[9] Further, under certain conditions on agents' preferences,[10] this mechanism also maximizes the number of potential matches.

In the presence of incomplete information, these results do not hold any more. The mechanism may has many equilibria depending on the available information about student abilities and the grade of coordination among colleges. In this section, we focus on two "natural" equilibria of the problem, with and without coordination among colleges, whose characterization permit us to analyze the effects of the presence of incomplete information and the problem of coordination in decentralized college admissions. The explicit characterization of agents' payoffs based on the number of students and colleges; college qualities and student skills allows us to identify the effect of a change in one of these parameters on equilibrium payoffs.

Before analyzing these equilibria, we argue that it is easy to characterize the equilibrium students' behavior. Since college qualities are observable, at any possible equilibria students must choose the best offer among the available ones. It is clear that under any alternative choice rule, students cannot get a better assignment. Then the rule where students choose the best offer among the available ones is a dominant strategy for students. We assume that colleges anticipate this optimal students' behavior and decide their offers. For simplicity, we label each student with one number from 1 to N. These labels are observable for all agents and do not provide information about student skills.

We analyze an equilibrium situation where colleges coordinate their actions based on students' labels. Consider a profile of strategies where students follow their dominant strategy while each college c_j sends one message to the student j. Let μ be the outcome matching of this strategy profile. It is easy to verify that this assignment satisfies $\mu(c_j) = j$ for $j = 1, ..., M$ while the rest of students remain unmatched, i.e. $\mu(j) = j$ for $j = M+1, ..., N$. Under this assignment, each college gets a payoff equal to $E[\alpha]$ while students get a payoff equal to v_j for $j = 1, ..., M$ and zero otherwise. It is easy to show that this profile of strategies is a SPE of this college admissions game. First of all, note that no student has a profitable deviation, since students are following their dominant strategy. Secondly, a college c_k can deviate by sending a message to any student $j \neq k$. In this case, the college c_k either will get matched with any student $j = k+1, ..., N$ or will be rejected by any student $j = 1, ..., k-1$. This deviation cannot be profitable, since all students are ex-ante identical. Note that under this equilibria, it is maximized the number of potential matches. This equilibrium is well defined for any permutation of the set of students and further these equilibria are payoff equivalent for colleges.

Now we consider an equilibrium of this game with no coordination among colleges. We want to show that the profile of strategies where students choose the best offer among the available ones and colleges make one offer to each student with equal probability is a SPE of this college admissions game.

We consider a college admissions problem with $M \geq 1$ colleges and $N \geq M$ students. As before, we label each student with one number from 1 to N with no information about student skills. Assume that each college sends one offer to each student with probability

[9] These results hold in problems where colleges have quotas of students providing colleges' preferences are responsive, see Roth and Sotomayor (1991).

[10] When colleges have responsive preferences respect to individual preferences, the set of agent unmatched and unfilled positions are the same at any stable matching (Roth and Sotomayor, 1990). This result implies that this simple matching mechanism not only exhausts the possibility of matching agents in a stable way, but also maximizes the number of potential matches when colleges have responsive preferences.

(i.e. $\frac{1}{N}$), we want to show that no college has a profitable deviation from this strategy. Consider that any college c_j with observable quality v_j is planning to deviate from this strategy. Note that there are $j-1$ colleges with higher quality and $M-j$ colleges with lower quality than the college c_j. Since college qualities are observable, an offer of the college c_j always beats any offer of the $M-j$ lower quality colleges. Then one offer of the college c_j will be accepted by a student i whenever every of the $j-1$ higher quality colleges make an offer to any of the other $N-1$ students.

The total number of combinations of offers from $M-j$ colleges to N students is N^{M-j} while total number of combinations of offers from $j-1$ colleges to $N-1$ students is $(N-1)^{j-1}$. Since colleges do not coordinate, one offer of the college c_j will be accepted by the students i with probability:

$$\frac{(N-1)^{j-1}\,N^{M-j}}{N^{M-1}} = \left(\frac{N-1}{N}\right)^{j-1} \quad \text{for } j = 1, ..., M. \tag{3}$$

Then by making an offer to the student i with probability $\frac{1}{N}$, the college c_j will get an expected payoff $\left(\frac{N-1}{N}\right)^{j-1} E[\alpha]$. Note that this payoffs are independent of student skills, since all students are ex-ante identical.

Now consider that the college c_j is planning to deviate from this strategy by making one offer to each student i with probability $p_i \neq \frac{1}{N}$. It is easy to see that such deviations cannot be profitable, since $\sum_{i=1}^{N} p_i^* \left(\frac{N-1}{N}\right)^{j-1} E[\alpha] = \sum_{i=1}^{N} p_i \left(\frac{N-1}{N}\right)^{j-1} E[\alpha]$ for any $p_i \neq \frac{1}{N}$ such that $\sum_{i=1}^{N} p_i = 1$. Then the profile of strategies where colleges send one offer to each student with equal probability is a symmetric SPE of this game. In this equilibrium, colleges' payoffs EQ_{c_j} depend on the number of students, the expected value of student skills and the rank of colleges.

$$EQ_{c_j} = \left(\frac{N-1}{N}\right)^{j-1} E[\alpha] \text{ for } j = 1, ..., M. \tag{4}$$

We deduce students' payoffs in this symmetric equilibrium. In this case, we have to find the probability that each student $i = 1, ..., N$ enrolls at college c_j for $j = 1, ..., M$. First of all, we know that the student i will reject any available offer but the best one. This implies that a student i enrolls at the college c_1 with probability $\frac{1}{N}$. It is easy to show that , in general, a student i enrolls at the college c_j with probability, $\frac{1}{N}\left(\frac{N-1}{N}\right)^{j-1}$. Then the expected payoff of the student $i = 1, ..., N$ is given by,

$$EU(N, M) = \frac{1}{N}\sum_{k=1}^{M} v_k \left(\frac{N-1}{N}\right)^{k-1} \tag{5}$$

Since students enroll at each college with positive probability, the student payoffs are strictly positive for any $M \geq 1$ and $N \geq M$ and satisfies $v_1 > EU(N, M) > 0$. In addition, students may remain unmatched with positive probability equal to $1-\frac{1}{N}\sum_{k=1}^{M}\left(\frac{N-1}{N}\right)^{k-1} = \left(\frac{N-1}{N}\right)^{M}$, which is strictly positive and increasing in the number of students and decreasing in the number of school places.

This simple model is useful to analyze the main consequences of the absence of coordination in college admissions with incomplete information. First of all, for any number of students N and school places M, all agents remain unmatched with positive probability but the highest quality college. Note that c_1 fills its vacancy with probability one and gets a expected payoff equal to $E[\alpha]$ which is the best prediction of student skills without additional information. Second, the equilibrium assignment may be inefficient, since colleges only know the expected value of student skills. Further, the probability of enrolling a students is decreasing in the rank of colleges, since the probability $\left(\frac{N-1}{N}\right)^{j-1}$ is strictly decreasing in j. Therefore, the absence of coordination mainly damages low quality colleges.

4 The costly signaling mechanism.

We analyze a decentralized matching mechanism called **Costly Signaling Mechanism** (CSM). Under this mechanism, each student $s \in S$ chooses a costly observable score $P_s \geq 0$ to signal his skills. A student $s \in S$ with type α who chooses a score P_s has to pay the cost

$$C(\alpha, P_s) = \frac{c(P_s)}{\phi(\alpha)} \tag{6}$$

We assume that the function $c(\cdot)$ such that $c(0) = 0$ is strictly increasing, continuous differentiable and convex. We also consider that the function $\phi(\cdot)$ such that $\phi(0) > 0$ is strictly increasing, continuous differentiable and bounded in the interval $[0, w]$.

The profile of student scores $(P_s)_{s \in S}$ is observable for all agents. Under the CSM, colleges and students are match according to the following decentralized matching procedure in two stages:

1. **Signaling Stage:** Each student $s \in S$ with parameter α chooses a score $P_s \geq 0$ at the cost $C(\alpha, P_s)$.

2. **Matching Stage:** After observing the profile of scores $(P_s)_{s \in S}$, students and colleges match according to the following decentralized matching process:

    (a) **Offers:** Each college $c \in C$ sends one message $m(c) \in S \cup \{c\}$. If $m(c) = s$, the college c is making an offer to the student s. If $m(c) = c$, the college c is making no offer. Let $O(s) = \{c \in C : m(c) = s\} \cup \{s\}$ be the set of offers of the student s (a student always receives an offer from himself) ;

    (b) **Hiring:** Each student $s \in S$ chooses one of his available offers $O(s)$.

We focus on a symmetric and strictly separating equilibrium where all students use the same signaling strategy. Obviously, the model can have many other symmetric equilibria. For instance, pooling equilibria where no student invests in signaling (in this situation we could sustain any of the symmetric equilibria analyzed in the previous section) or semi-separating equilibria.

To sustain this separating equilibria, we consider beliefs where the higher scored students are associated with higher academic skills. Formally, we describe these beliefs by a continuous distribution of student skills given a score $P > 0$, i.e. a continuous cumulative distribution $G(\alpha \mid P)$. We assume that these beliefs have associated a continuous density $g(\alpha \mid P)$ and satisfy $G(\alpha \mid P') < G(\alpha \mid P)$ for all $\alpha \in (0, w)$ whenever $P' > P$. Note that the previous condition implies that $E[\alpha \mid P'] > E[\alpha \mid P]$ for all $\alpha \in (0, w)$ whenever $P' > P$ where $\int \alpha g(\alpha \mid P)\, d\alpha = E[\alpha \mid P]$. This implies that higher scored students are associated with higher expected skills, i.e. colleges prefer high scored students.

The payoffs of colleges are given by the expected quality of enrollees that depends on the outcome of the CSM (i.e. a matching between colleges and students) and the profile of student scores. Given a matching μ, a college $c \in C$ has expected payoff $E\left[\alpha \mid P_{\mu(c)}\right]$. On the other hand, a student $s \in S$ with parameter α gets payoffs $v_c - C(\alpha, P_s)$ such that $\mu(s) = c$ and $-C(\alpha, P_s)$ otherwise.

In order to simplify, we consider that after observing the profile of student scores $(P_s)_{s\in S}$, colleges "update" their ordinal preferences in the following way. Each college $c \in C$ forms an auxiliary preference relation $\succ_c^*$ over the set students and the prospect of remaining unmatched, $S \cup \{c\}$ such that: a) $s \succ_c^* c$ if and only if $P_s > 0$ and b) for any $s, s' \in S$, $s \succ_c^* s'$ if and only if $P_s > P_{s'}$. This profile of interim preferences$\succ_C^* = (\succ_c^*)_{c\in C}$ is consistent with beliefs $G(\alpha \mid P)$.

At the second stage of the CSM, the profile of student scores is given, so colleges form the interim preference $\succ_C^* = (\succ_c^*)_{c\in C}$ while the the signaling cost is sunk for students. This implies that student preferences at this stage are well defined and coincide with the strict preferences $\succ_S = (\succ_s)_{s\in S}$. Assume w.l.g. that colleges have different qualities, i.e $v_1 > v_2 > .. > v_M > 0$. On the other hand, assume also that students' scores satisfy $P_s \neq P_{s'}$ for all $s, s' \in S$. This assumption about student scores is not strong, since in equilibrium ties will happen with probability zero, i.e. no pair of students will have the same skills. Then for any given profile of student scores $(P_s)_{s\in S}$, at the second stage of the CSM there is a well defined college admissions problem with strict preferences denoted by $(S, C, (\succ_S, \succ_C^*))$.

Note that given the beliefs $G(\alpha \mid P)$, all colleges and students have identical and strict preferences. Then the set of stable matchings of this problem is not empty and single value, i.e. $|\mathcal{E}(S, C, (\succ_S, \succ_C^*))| = 1$. Under these conditions, we can establish the following result.

Proposition 1 *Consider the beliefs $G(\alpha \mid P)$ such that such that $G(\alpha \mid P') < G(\alpha \mid P)$ for all $\alpha \in (0, w)$ whenever $P' > P$. Then for any profile of student scores $(P_s)_{s\in S}$ such that $P_s \neq P_{s'}$ for all $s, s' \in S$ and $s \neq s'$, there is a unique SPE outcome in the second stage of the CSM. This equilibrium outcome is the unique stable matching of the college admissions problem, $(S, C, (\succ_S, \succ_C^*))$. Further, this unique equilibrium assignment is assortative.*

Only stable matchings between students and colleges are a reasonable outcomes of the CSM (Alcalde and Romero-Medina, 1998). Further, under the interim college preferences $\succ_C^* = (\succ_c^*)_{c\in C}$ the outcome matching is assortative, i.e. the highest scored student is matched with the best college; the second highest scored student is matched with the second best college, and so on.

In the following section, we analyze the signaling equilibrium of the first stage of the CSM. We focus on a symmetric pure strategy equilibrium where all students play according to the same signaling strategy. We analyze a settings with $M \geq 1$ school places and $N > M$ students. However, the model can be easily extended to analyze any problem with the same number of students and school places.

4.1 The signaling equilibrium

We analyze the signaling equilibrium of the first stage of the CSM. To characterize this equilibrium, we take as given the outcome matching of the second stage of the mechanism.

To illustrate the problem, we focus on the simplest case with only one college with quality $v_1 > 0$ and $N > 1$ students.

We analyze a separating symmetric Nash equilibrium of the signaling game played by students. This equilibrium is characterized by a continuous differentiable and strictly increasing signal strategy with respect to student abilities. We focus on the student 1's problem, who chooses a score P_1 to signal his skills while the rest of students play according to a signaling strategy $\rho : [0, w] \to \Re_+$ which is assumed to be strictly increasing and continuous differentiable in α such that $\rho(0) = 0$.

Since the outcome matching of the CSM is assortative, the students 1 with parameter α gets the payoff $v_1 - C(\alpha, P_1)$ whenever $P_1 > \rho(\alpha_i)$ for all $i \neq 1$. This happens with probability $\Pr[P_1 > \rho(\alpha_2), ..., P_1 > \rho(\alpha_N)] = F\left(\rho^{-1}(P_1)\right)^{N-1}$ given that student skills are identically and independently distributed according to F. Otherwise, the students 1 gets the payoff $-C(\alpha, P_1)$ with probability $1 - F\left(\rho^{-1}(P_1)\right)^{N-1}$. Hence, the expected payoffs of the student 1 with parameter α, when the rest of students play according to the signal function $\rho(\cdot)$ is:

$$\pi(\alpha, P_1) = v_1 F\left(\rho^{-1}(P_1)\right)^{N-1} - C(\alpha, P_1) \tag{7}$$

The student 1 takes as given the signaling strategy of the rest of students and chooses a score P_1 to maximize his expected payoff $\pi(\alpha, P_1)$. The first order condition (FOC) with respect to P_1 yields the following equation,

$$v_1 (N-1) F\left(\rho^{-1}(P_1)\right)^{N-2} f\left(\rho^{-1}(P_1)\right) \frac{1}{\rho'\left(\rho^{-1}(P_1)\right)} - \frac{c'(P_1)}{\phi(\alpha)} = 0 \tag{8}$$

By reordering the previous expression, we have the following differential equation,

$$v_1 (N-1) \phi(\alpha) F\left(\rho^{-1}(P_1)\right)^{N-2} f\left(\rho^{-1}(P_1)\right) = c'(P_1) \rho'\left(\rho^{-1}(P_1)\right) \tag{9}$$

In a symmetric equilibrium, $P_1 = \rho(\alpha)$ then the previous differential equation becomes in the following,

$$v_1 (N-1) \phi(\alpha) F(\alpha)^{N-2} f(\alpha) = c'(\rho(\alpha)) \rho'(\alpha) \tag{10}$$

By solving this differential equation with the initial condition $\rho(0) = 0$, we find that the equilibrium signaling strategy satisfies,

$$\rho(\alpha) = c^{-1}\left(v_1 (N-1) \int_0^\alpha \phi(x) F(x)^{N-2} f(x)\, dx\right) \tag{11}$$

The equilibrium signaling strategy $\rho(\cdot)$ is strictly increasing and continuous differentiable in α. Note that $\rho(\cdot)$ only satisfies the FOC of the student 1's maximization problem, which is necessary but not sufficient to characterize the signaling equilibrium. Hence, we have to prove that the signaling strategy $\rho(\cdot)$ is in fact a symmetric equilibrium of this game. The equilibrium payoff of any student with parameter α is given by,

$$\pi(\alpha, \rho(\alpha)) = v_1 F(\alpha)^{N-1} - \frac{c(\rho(\alpha))}{\phi(\alpha)} \tag{12}$$

It is not difficult to show that this payoff function satisfies $\frac{d}{d\alpha}(\pi(\alpha, \rho(\alpha))) > 0$ and $\pi(0, \rho(0)) = 0$. We show that any alternative score $P' \neq \rho(\alpha)$ cannot be a profitable deviation for any student with parameter α. Consider that a student with parameter α is planning to choose a score $0 < P' < \rho(\alpha)$ while the rest of students are playing according to the signaling strategy $\rho(\alpha)$. Since the signaling strategy is strictly increasing in α and satisfies $\rho(0) = 0$, there exists a unique α' such that $\rho(\alpha') = P'$. This implies that a student who chooses an alternative strategy $P' = \rho(\alpha')$ will get a payoff, $\pi(\alpha, P') = \pi(\alpha, \rho(\alpha'))$. A student with parameter α losses an extra payoff $\pi(\alpha, \rho(\alpha)) - \pi(\alpha, \rho(\alpha'))$ by not deviating to $\rho(\alpha') = P'$. Then

$$\pi(\alpha, \rho(\alpha)) - \pi\left(\alpha, \rho\left(\alpha'\right)\right) = v_1 \left(F(\alpha)^{N-1} - F\left(\alpha'\right)^{N-1}\right) - \frac{c(\rho(\alpha)) - c(\rho(\alpha'))}{\phi(\alpha)} \tag{13}$$

It is not difficult to show that the extra payoffs $\pi(\alpha, \rho(\alpha)) - \pi(\alpha, \rho(\alpha'))$ can be reduced to the following expression,

$$v_1 \left(F(\alpha)^{N-1} - F\left(\alpha'\right)^{N-1}\right) - \frac{1}{\phi(\alpha)} v_1 (N-1) \int_{\alpha'}^{\alpha} \phi(x) F(x)^{N-2} f(x)\, dx \tag{14}$$

Since the function $\phi(x)$ is always positive, strictly increasing in x and bounded in $[0, w]$, it is clear that the following inequality holds,

$$\frac{1}{\phi(\alpha)} v_1 (N-1) \int_{\alpha'}^{\alpha} \phi(x) F(x)^{N-2} f(x)\, dx < v_1 \int_{\alpha'}^{\alpha} (N-1) F(x)^{N-2} f(x)\, dx \tag{15}$$

In addition note that $\int_{\alpha'}^{\alpha} (N-1) F(x)^{N-2} f(x)\, dx = F(\alpha)^{N-1} - F(\alpha')^{N-1}$. This last equation implies that $\pi(\alpha, \rho(\alpha)) - \pi(\alpha, \rho(\alpha')) > 0$ for all $\alpha' < \alpha$ which proves that $P' = \rho(\alpha')$ is not a profitable deviation. By a similar argument, it is possible to show that any alternative score $P'' > \rho(\alpha)$ cannot be a profitable deviation. Then the signaling strategy $\rho(\alpha)$ is a symmetric equilibria of the signaling game played by students. In the following section, we show that all of these results hold in the general case with $M \geq 2$ colleges and $N > M$ students. All proofs and calculations can be found in the Appendix.

4.1.1 The general case: $N > M \geq 2$.

Now consider a general case with N students and M colleges such that $N > M \geq 2$. Assume w.l.g. that all colleges have different qualities and satisfy $v_1 > v_2 > ... > v_M > 0$. As before, we analyze the student 1's maximization problem with parameter $\alpha \in (0, w)$ while all other students play according to some signaling strategy $\rho_M : [0, \infty) \to \Re_+$. As before, we assume that the signaling strategy $\rho_M(\cdot)$ is strictly increasing and continuous differentiable in α such that $\rho_M(0) = 0$.

The student 1 chooses a score $P_1 \geq 0$ to signal his abilities. Consider the following notation, we say that the student 1 has a "success" whenever $P_1 > \rho_M(\alpha_i)$ for some student $i \neq 1$ and a "failure" whenever $P_1 < \rho_M(\alpha_i)$ for some student $i \neq 1$. The probability of having one "success" is $F\left(\rho_M^{-1}(P_1)\right)$ whereas the probability of having one "failure" is $1 - F\left(\rho_M^{-1}(P_1)\right)$. Note that these probabilities are independent, since students' parameters are independently distributed.

For any given number of students $N \geq M$, the student 1 with score P_1 enrolls at the colleges c_j with quality v_j, whenever he has $N - j$ "successes" and $j - 1$ "failures". Note that this situation happens $\binom{N-1}{j-1}$ different times, then the probability of enrolling at the college c_k is,

$$\binom{N-1}{k-1} F\left(\rho_M^{-1}(P_1)\right)^{N-k}\left[1-F\left(\rho_M^{-1}(P_1)\right)\right]^{k-1}. \tag{16}$$

The previous argument implies that the probability of enrolling at the college $c_j \in C$ follows a binomial distribution. Hence, the expected payoff of the student 1 $\pi(\alpha, P_1)$ satisfies,

$$\pi(\alpha, P_1) = \sum_{k=1}^{M} v_k \binom{N-1}{k-1} F\left(\rho_M^{-1}(P_1)\right)^{N-k}\left[1-F\left(\rho_M^{-1}(P_1)\right)\right]^{k-1} - C(\alpha, P_1) \tag{17}$$

The student 1 takes as given the signaling strategy of the rest of students and chooses a score P_1 to maximize his expected payoff $\pi(\alpha, P_1)$. In Appendix A, we solve the student 1's maximization problem in a symmetric equilibrium where all students play according to the same signaling strategy $\rho_M(\alpha)$. We show that the signal function that satisfies the FOC of the student 1's maximization problem characterizes this symmetric separating equilibrium. We establish the following result,

Proposition 2 *The signaling strategy,*

$$\rho_M(\alpha) = c^{-1}\left(\sum_{k=1}^{M-1}(v_k - v_{k+1})\int_0^\alpha \phi(x) f_{(k,N-1)}(x)\,dx + v_M \int_0^\alpha \phi(x) f_{(M,N-1)}(x)\,dx\right)$$

is a symmetric equilibrium of college admissions problems with $M \geq 2$ colleges and $N > M$ students.

Proof. See Appendix A. ■

Given the equilibrium signaling strategy $\rho_M(\cdot)$, a student with parameter α will get expected payoffs,

$$\pi(\alpha, \rho_M(\alpha)) = \sum_{k=1}^{M} v_k \binom{N-1}{k-1} F(\alpha)^{N-k}\left[1-F(\alpha)\right]^{k-1} - \frac{c(\rho_M(\alpha))}{\phi(\alpha)} \tag{18}$$

Note that to characterize the signaling equilibrium, we assume some desirable properties of the equilibrium signaling strategy. We should show that these properties are satisfied in equilibrium. A simple observation is enough to show that the equilibrium signaling strategy and agents' payoff are continuous differentiable functions in α. In the following result, we establish some interesting properties of these signaling strategy and equilibrium payoffs.

Proposition 3 *The equilibrium signaling strategy $\rho_M(\alpha)$ and student payoffs $\pi(\alpha, \rho_M(\alpha))$ satisfy the following properties:*

1. *$\rho_M(\alpha)$ is strictly increasing in α and bounded above.*
2. *$\pi(\alpha, \rho_M(\alpha))$ is strictly increasing in α.*

Proof. See Appendix A. ■

Since the equilibrium signaling strategy is strictly increasing and probability of having two students with the same skills is zero, no pair of students choose the same score to signal their skills. Then the equilibrium outcome of the CSM will be assortative with respect to the true student skills. Hence, the highest skilled student will be matched with the best college; the second highest skilled student will be matched with the second best college; and so on. Further, students with higher abilities get greater payoffs. This result comes from the single crossing property of the signaling cost, since higher skill students have lower marginal signaling cost.

On tho other hand, the assortative structure of the equilibrium assignment of the CSM implies that colleges payoffs depend on the ranking of enrollees and the prior distribution of student skills. Let μ^* be the unique equilibrium outcome of the CSM, then in equilibrium colleges get expected payoffs,

$$EQ^*_{c_j} = E\left[\alpha \mid P_{\mu^*(c)}\right] = E\left[\alpha_{(j,N)}\right] = \int_0^w \alpha f_{\alpha_{(j,N)}}(\alpha)\, d\alpha \text{ for } j = 1, ..., M. \tag{19}$$

Where $\alpha_{(j,N)}$ is the j-th order statistic from a sample of size N such that $\alpha_{(1)} = \max\limits_{1\leq i\leq N} \alpha_i$, $\alpha_{(2)}$ = second greatest α_i, and so on. It is well known that the order statistic $\alpha_{(j,N)}$ is distributed according to the probability density function,

$$f_{\alpha_{(j,N)}}(\alpha) = \frac{N!}{(j-1)!\,(N-j)!} f(\alpha)\, F(\alpha)^{N-j} \left[1 - F(\alpha)\right]^{j-1} \text{ for } j = 1, ..., M. \tag{20}$$

Under this conditions, it is not difficult to show that responding to students' signals is an equilibrium situation for colleges. First of all, it is not difficult to show that the best strategy for any college c_j is to respond to students' signals providing all higher quality colleges $\{c_1, c_2, ..., c_{j-1}\}$ do. The argument is very simple, college c_j has to compare the expected skills of enrollees between responding to students' signals and any alternative admission rule. Note that c_j knows that all students are willing to accept its offer but those already enrolled at colleges $\{c_1, c_2, ..., c_{j-1}\}$, since by assumption those colleges respond to signals and have greater qualities. This implies that any potential enrollee of the college c_j has skills $\alpha \leq \alpha_{(j)}$. By responding to students' signals, the college c_j will enroll the best student among the available ones. In contrast, with any other admission rule it will enroll a lower skilled students.

Now consider the case of the best college, c_1 knows that its offer will be accepted by any student. Since by responding to students' signals, c_1 will enroll the best student among all students, it optimally responds to students' signals. Then a simple induction argument shows that all colleges respond to students' signals.

5 Comparative statics.

In previous sections, we analyze a separating symmetric equilibrium of the signaling game induced by the CSM. We characterize an equilibrium signaling strategy that depends on several parameters like the prior distribution of skills; the number of students and school seats; and college qualities. Our explicit characterization allows us to conduct interesting comparative statics exercises to analyze the impact of a change in the underlying parameters of the model on the equilibrium signaling strategy. In particular, we focus on three experiments:

1. A change in the number of students;

2. A change in the number of school places; and

3. A change in the quality of colleges.

The analysis of these experiments is crucial to understand real-world colleges admissions as a signaling process whose outcome depends on the interaction of strategic decision makers. Our model provides a good approach to analyze the effect of a change of these underlying parameters given our closed form characterization of the signaling equilibrium and empirical evidence that support our theoretical results.

One of the most important real-world signal in college admissions is the SAT test in the US. Most students take the SAT during the last year of high school, and almost all colleges and universities use it to make admission decisions. Several empirical studies analyze the importance of the SAT and provide empirical evidence that support our model of decentralized college admissions with incomplete information and costly signaling. First of all, the SAT is a costly signal that depends on on the amount of effort invested by students. Further, according to the structure and content of the SAT is expected that higher skilled students get better scores. On the other hand, empirical evidence suggests a strong correlation between SAT scores and student skills. For instance, there is a high correlation between SAT scores and several measures of student success like IQs and the probability of success in college (Frey and Detterman, 2004). Finally, it is observed that the best colleges and universities tend to enroll students with high SAT scores (Webster, 2001a, 2001b). This implies that the matching between colleges and students tend to be assortative with respect to the true student skills.

A bad understanding of the underlying signaling game in college admissions may lead us to suggest wrong policy recommendations. For instance, empirical evidence in the US shows a decline in the mean SAT scores as the participation rate increases. If we only consider the high correlation between SAT scores and measures of student skills. We can suggest that the decline in SAT scores comes from an increase in the proportion of low skilled students, i.e. a change in the current distribution of student skills. According to this argument, a policy recommendation would be to increase the expenditure in SAT coaching and tutorials in high school in order to improve abilities of new test-takers. However, the previous argument and policy recommendation ignore the signaling game in college admissions, since strategic students may decrease the investment in signaling in the face of new competitors with no change in the distribution of student skills.

5.1 A change in the number of students.

We analyze the effect of a change in the number of students over the equilibrium signaling strategy $\rho_M(\alpha) = \rho_M(\alpha, N)$. Intuitively, an increase in the number of students should increase the competition for school places. It seems reasonable that this increase in competition should lead students to increase the investment in signaling in order to beat new competitors. This intuition seems correct, however the effect of increasing the competition may not be symmetric among students. The probability of beating potential competitors depends on student abilities. A low skilled student knows that the probability of facing new high skilled competitors is big while a high skilled student knows that the probability of facing qualified competitors is small.

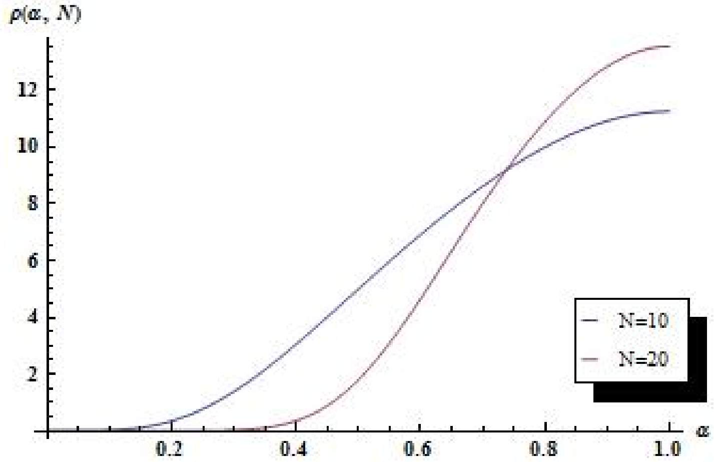
Our characterization of the equilibrium signaling strategy allows us to conduct a comparative static exercise where we increase the number of students while we maintain fixed all other parameters of the model. The result of this exercise shows that this effect depends on academic abilities. Formally,

Proposition 4 *For any college admissions problem with $M \geq 1$ colleges and $N > M$ students, $\rho_M(\alpha, N+1) < \rho_M(\alpha, N)$ for all $\alpha \leq \alpha_N(N)$ and $N > M$. Further, the threshold $\alpha_N(N)$ is strictly monotone increasing N.*

Proof. See Appendix B. ■

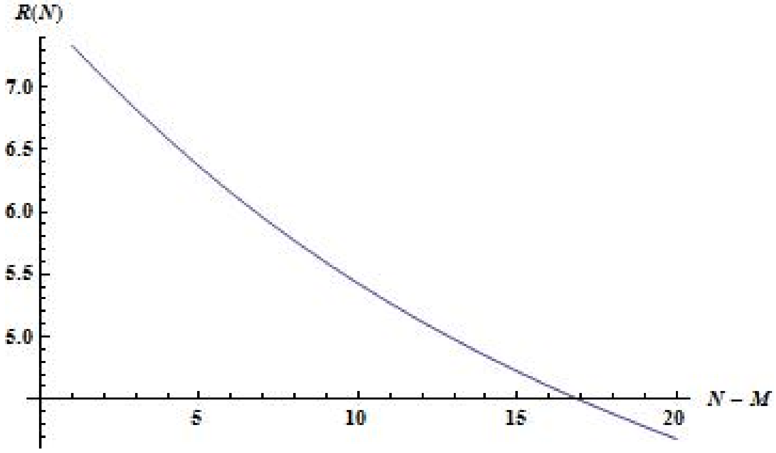
The previous result has two main implications. First of all, we find that low qualified students decrease the investment in signaling while the high skilled students may increase it, as the number of students increases. Intuitively, a low skilled student should beat $N-k$ competitors to get a place at college c_k. When the number of students increases to $N' > N$, he not only should beat $N'-k$ competitors to get the same school place but also has a big probability of facing new high skilled competitors. In contrast, a high skilled student should also beat more competitors to enroll at college but at the same time he has a small probability of facing new high skilled competitors. These two opposite effects may lead high skilled students to increase the probability of enrolling at college and the investment in signaling.

Figure 1: Effect of increasing the number of students



The second interesting implication regards with the monotonicity of the threshold $\alpha_N(N)$. We find that this threshold is monotone increasing in N, i.e. $\alpha_N(N+1) > \alpha_N(N)$ for any $N > M$. This implies that students do not increase the investment in signaling when they face $N+1$ competitors if they have already decreased it with N. This property of the equilibria and the fact that the the equilibrium signaling strategy is bounded above allow us to infer the evolution of the average investment in signaling when the number of students increases. Let $R(N) = \int \rho_M(\alpha, N) f(\alpha) d\alpha$ be the expected (average) investment in signaling, then there should exist a sufficiently large demand for school places $\hat{N}$ such that $R(N+1) < R(N)$ for all $N \geq \hat{N}$.

Figure 2: Average investment in signaling with respect to N



Our results are consistent with an interesting real-world fact. In the US college admissions system, it has been extensively analyzed the impact of increasing the number of test-takers on the mean SAT scores. According to data of the College Board for several years, it has been observed a decline in the mean SAT scores as the participation rate increases. The College Board explains this stylized fact in the following way[11]:

"*It is common for mean scores to decline slightly when the number of students taking an exam increases because more students of varied academic backgrounds are represented in the test-taking pool.*"

On the one hand, this interpretation only considers the positive correlation between the SAT and students abilities to conclude that an increase in the number of applicants systematically decreases the proportion of good test-takers. On the other hand, this interpretation does not consider the possibility that students optimally change their strategies in the face of new competitors. Clearly, this explanation ignores the underlying signaling game in college admissions and tries to justify that a change in the number of competitors affect the distribution of student skills. In contrast our model suggests that an increase in the number of applicants not only leads low skilled students to decrease the investment in signaling but also increases the proportion people who reduce it. Then an increase in the number of competitors will eventually reduce the average investment in signaling with no change in the prior distribution of student abilities.

5.2 A change in the number of school places.

The following experiment regards with the effect of changing the number of school places on the equilibrium signaling strategy. We consider an experiment where the number of school places can increase but remaining lower than the number of students. Intuitively, there is decrease in competition for school places that should lead students to reduce the investment in signaling

Our model shows that this intuitive argument may not be correct, at least not for all students. As in the previous case, the effect of a change in the number of school places is not symmetric across student. In order to simplify, we consider a very simple model with

[11]College board (2011), "43% of 2011 College-Bound Senior Met SAT College and Career Readiness Benchmark" at http://press.collegeboard.org/releases/2011/43-percent-2011-college-bound-seniors-met-sat-college-and-career-readiness-benchmark

N students and one college that offers $M \geq 1$ school seats. In this setting, it is easy to show that the equilibrium signaling strategy of the problem satisfies the following,

$$\rho_M(\alpha, M) = c^{-1}\left(v_1 \int_0^\alpha \phi(x) f_{(M,N-1)}(x)\, dx \right) \tag{21}$$

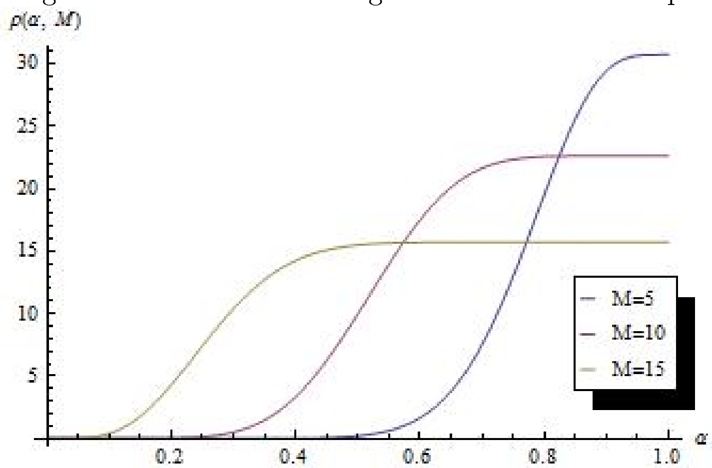
Then we establish the following result,

Proposition 5 *For any college admissions problem with one college with $M \geq 2$ school seats and $N > M+1$ students, $\rho_M(\alpha, M+1) > \rho_M(\alpha, M)$ for all $\alpha \leq \alpha_M(M, N)$. Further, the threshold $\alpha_M(M, N)$ is monotone increasing in N and monotone decreasing in M.*

Proof. See Appendix B. ■

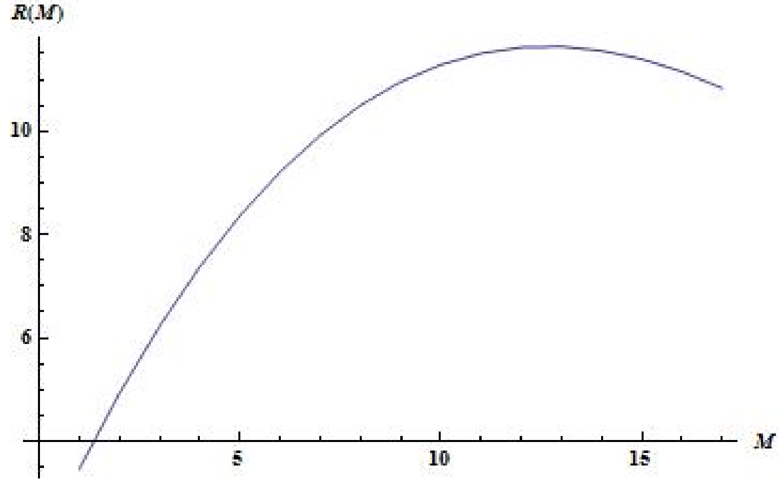
The previous result has several interesting implications. First of all, an increase in the number of school places leads low skilled students to increase the investment in signaling while the high qualified students may decrease it. Intuitively, an increase in the number of school places should be equivalent to a decrease in the number of students with a fixed number of school seats. When a student leaves the market, the low qualified students increase their probability of enrolling at college, since they should beat fewer competitors and the probability of facing high skilled competitors decreases. This increment in the probability of enrolling at college leads low skilled students to increase the investment in signaling. On the other hand, the best students face a decrease in the probability of enrolling at college. Then for high qualified student is more likely to face another high skilled competitor as the number of students decreases. This increase in competition lead the high skilled students to decrease the investment in signaling.

Figure 3: Effect of increasing the number of school places



We also show that threshold $\alpha_M(M, N)$ is monotone decreasing in M, i.e. $\alpha_M(k+1, N) < \alpha_M(k, N)$ for $k = 1, ..., M$. This result allows us to establish some general conclusions about the evolution of the average investment in signaling as a function of the number of school places. We prove that low and high skilled students change the investment in signaling in opposite directions. When there are a few available school seats, a new one is very valuable and leads students to increase the average investment in signaling. However, when the number of school places increases, the proportion of people that decrease the investment also increases, since a new school place is less valuable. Then, there should be a big enough number of school places, from which an additional school seat decreases the average investment in signaling.

Figure 4: Average investment in signaling with respect to M



5.3 A change in the quality of colleges

In this section, we analyze the effect of a change in college qualities on the equilibrium signaling strategy. We focus on a change in qualities that preserve ordinal student preferences. For instance, if the college c_k changes its quality from v_k to v'_k, it should be true that $v_{k-1} > v'_k > v_{k+1}$, whenever $v_{k-1} > v_k > v_{k+1}$. This assumption makes comparable the equilibrium signaling strategies before and after the change in college qualities, since the equilibrium assignment of the CSM is the same in both situations.

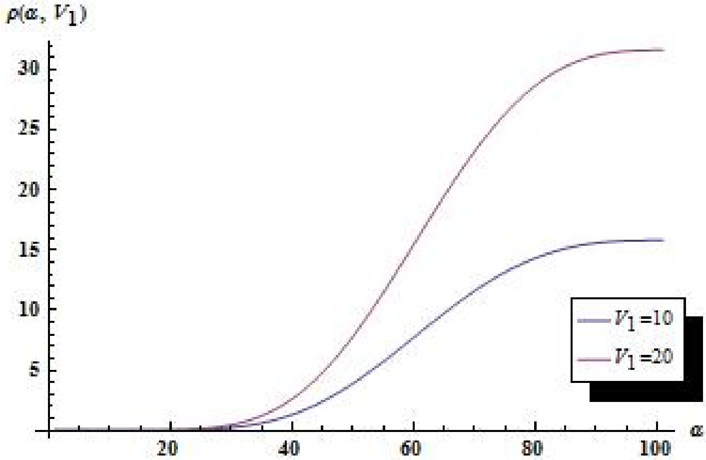
Intuitively, when a college increases its quality the average quality of schools also increases, this increment in students' valuations makes reasonable to increase the investment in signaling. However, as in the previous cases, this result depends on student abilities. Let $sgn\,(x)$ be a function such that $sgn\,(x) = 1$ if $x > 0$, $sgn\,(x) = -1$ if $x < 0$ and $sgn\,(x) = 0$ if $x = 0$. Then we establish the following result.

Proposition 6 *For any college admissions problem with $M \geq 2$ colleges and $N > M$ students, $sgn\,(\rho_M\,(\alpha, v'_k) - \rho_M\,(\alpha, v_k)) = sgn\,(v'_k - v_k)$ for all $\alpha \leq \alpha_{v_k}\,(N, k)$ and $k = 1, ..., M$. Further, the threshold $\alpha_{v_k}\,(N, k)$ is monotone increasing N for all $k = 2, ..., M$ and monotone decreasing in k.*

Proof. See Appendix B. ■

The previous result has interesting implications. First of all, only low skilled students are willing to increase the investment in signaling while the high qualified students may decrease it. Intuitively, the an increase in college qualities is more valuable for low skilled students than for high skilled ones. This implies that only an increase in the quality of the best colleges leads the highest skilled students to increase the investment in signaling. On the other hand, we also show that the threshold $\alpha_{v_k}\,(N, k)$ is monotone decreasing in k, i.e. $\alpha_{v_k}\,(N, k+1) < \alpha_{v_k}\,(N, k)$ for all $k = 1, ..., M-1$. As expected, students are more willing to increase their investment for high quality colleges. Further, in Appendix B we show that $\alpha_{v_1}(N,1) = w$ for any N, which implies that only an increase in the quality of the college c_1 has no asymmetry across students, i.e. all students are willing to increase the investment in signaling.

Figure 5: A change in the quality of the best college



6 Gains of the CSM

In this paper, we analyze a separating symmetric equilibria of the CSM that maximizes the number of potential matches and lead the best students to enroll at the best colleges. In contrast with no signaling, low quality colleges are able to enroll better students with positive probability. Then some colleges may prefer to run an admissions system with no signaling to enroll better students with positive probability. A similar argument applies for low skilled students, whom pay the signaling cost and lose the chance of enrolling at high quality colleges.

According to the previous argument, some agents may get losses under the CSM in the sense that they can get better assignments and payoff with no signaling. Further, it seems that low quality colleges and low skilled students are the most damaged agents under the CSM. We define the gains of implementing the CSM in a natural way, as the difference in equilibrium payoffs between the separating signaling equilibria of the CSM and the symmetric equilibria of college admissions problem with no signaling. According to this definition, students' gains are defined as follows,

$$L(\alpha) = \pi(\alpha, \rho_M(\alpha)) - EU(N,M) \tag{22}$$

Since the student's payoff in the game with no signaling $EU(N,M)$ is type independent, students' gains are strictly increasing in α. However, there always exits a proportion of people that gets losses under the CSM, since $\pi(0, \rho_M(0)) = 0$ and $EU(N,M) > 0$. Note that eventually all students may get losses depending on the prior distribution of skills. However, only the highest skilled students have the possibility of getting positive gains.

The previous definition implies that college c_j's gains are defined as follows,

$$\Delta EQ(j,N) = EQ^*_{c_j} - EQ_{c_j} \text{ for } j = 1, ..., M. \tag{23}$$

Where $EQ^*_{c_j} = E\left[\alpha_{(j)}\right]$ is c_j's payoff in the separating equilibria of the CSM and $EQ_{c_j} = \left(\frac{N-1}{N}\right)^{j-1} E\left[\alpha\right]$ is c_j's payoff in the game with no signaling. The analysis of these gains is not a trivial issue. For instance, it is clear that with no signaling an increase in the number of applicants must increase the probability of enrolling a student with average abilities. Under the CSM, an increase in the number of applicants should lead colleges to enroll better and better students. Since colleges gains are the difference between these payoffs, it is not clear the final effect of increasing the number of students on colleges' gains. Similar arguments can be applied in the case of qualities, it is not clear which colleges get the highest gains or if colleges gains are monotone in college rankings.

On the other hand, to analyze colleges' gains we require the analysis of order statistics. This is a difficult problem, since most distributions have no closed form solutions for moments of order statistics. To analyze this problem, we consider particular prior distributions where it is possible to find closed form formulas for these moments. We focus on the exponential model[12], in this case it is possible to show that $E\left[\alpha_{(j)}\right] = \sum_{k=1}^{N+1-j} \frac{\theta}{N+1-k}$ for $j = 1, ..., M$ where $E\left[\alpha\right] = \theta$ (Huang, 1974). Then, colleges' gains $\Delta EQ\left(j, N\right)$ satisfy the following equation,

$$\Delta EQ\left(j, N\right) = \theta \sum_{k=1}^{N+1-j} \frac{1}{N+1-k} - \theta \left(\frac{N-1}{N}\right)^{j-1} \tag{24}$$

Then, we establish the following result.

Proposition 7 *Consider any M by N college admissions problem such that $N > M \geq 1$. Assume that students' skills are exponentially distributed with parameter $\theta > 0$. Then the following holds:*

1. *$\Delta EQ\left(j, N\right)$ are strictly monotone increasing in N, i.e. $\Delta EQ\left(j, N+1\right) > \Delta EQ\left(j, N\right)$ for all $N > M$ and $j = 1, ..., M$;*

2. *$\Delta EQ\left(j, N\right)$ are strictly monotone decreasing in j, i.e. $\Delta EQ\left(j+1, N\right) < \Delta EQ\left(j, N\right)$ for all $N > M$ and $j = 1, ..., M-1$; and*

3. *For any $M \geq 1$ there always exists an $N^* > M$ such that $\Delta EQ\left(j, N\right) \geq 0$ for all $j = 1, ..., M$ and all $N \geq N^*$.*

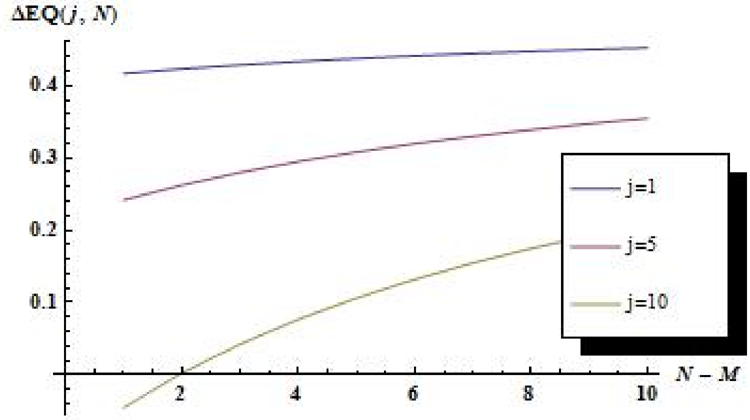
Proof. See Appendix C. ■

The previous result has interesting implications. First of all, an increasing demand for school places improve the gains of enrolling students based on costly signals. Intuitively, an increasing pool of students leads colleges to reduce the risk of remaining unmatched while a costly signal becomes more and more effective to pick the best available students. Another interesting implication regards with the comparison of gains among colleges. As in the case of students, colleges' gains can be ranked according to college qualities. This result implies that the big winners of the CSM are the high quality colleges, which not only enroll the best students but also get the greatest gains. The third interesting implication regard

[12] Skills are exponentially distributed with parameter $\theta > 0$, if α is distributed according the density function, $f\left(\alpha; \theta\right) = \frac{1}{\theta} e^{-\frac{\alpha}{\theta}}$. In this case, the cumulative distribution function is $F\left(\alpha; \theta\right) = 1 - e^{-\frac{\alpha}{\theta}}$. In addition, $E\left[\alpha\right] = \theta$.

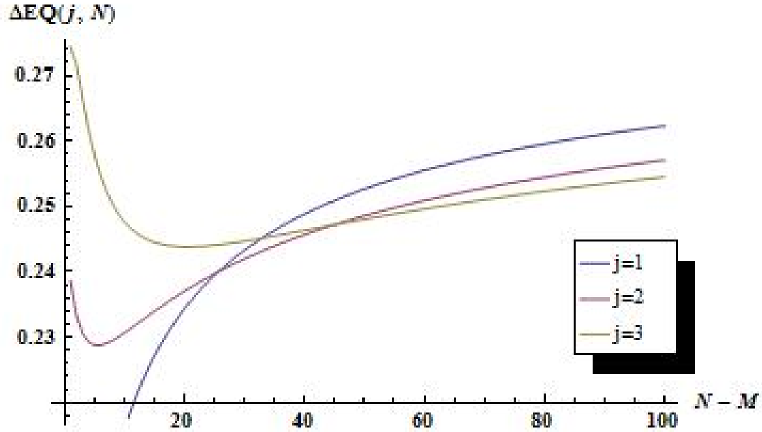
with the relationship between the size of the demand for school places and colleges' gains. We find that a big enough demand for school places leads all colleges to get positive gains. This result contrasts with the case of students, where there is a proportion of students that always get losses.

Figure 6: Colleges' gains with exponential distributed skills



It is easy to show that the previous results cannot be trivially extended to any prior distribution of student skills. As we show in the following figure, we cannot guarantee neither the monotonicity of colleges' gains with respect to the number of students nor the monotonicity with respect to colleges' qualities. In this case, we consider Beta distributed skills with parameters $a = 10$ and $b = 2$. Note that this distribution is skewed to the right, this fact may explain why the previous results about colleges' gains do not hold any more, since the probability of enrolling a good student with no signaling is significantly big.

Figure 7: Colleges' gains with Beta (10,2) distributed skills



7 Conclusion

We analyze some consequences of coordination problems in decentralized college admissions with incomplete information. We consider a matching problem where colleges with observable qualities want to enroll student whose abilities are private information. We analyze a simple decentralized matching mechanism called **Costly Signaling Mechanism** (CSM). Under the CSM, students choose a costly observable score to signal their skills. We characterize a separating symmetric equilibrium of the game induced by the CSM. In this equilibrium the CSM maximizes the number of potential matches of the problem and induces agents to matched efficiently, in the sense that the best students will enroll at the best colleges. Hence, for the case in which the number of students equals the number of school seats, all agents will get matched while when there are more students than school places only the highest skilled students will get matched.

We conduct three exercises of comparative statics that allow us to analyze the impact of a change in the underlying parameters of the model on the equilibrium signaling strategy. Our main result shows that this effect is not symmetric across students, since they depend on student abilities. The first comparative statics exercise regards with the effect of a change in the number of students. In this case, we show that an increase in the number of students leads low skilled students to decrease the investment in signaling while the high skilled students may increase it. We also analyze the effect of a change in the number of school places and a change in college qualities with similar implications.

Finally, we analyze the gains of the CSM which are defined in a natural way as the difference in equilibrium payoffs between the separating signaling equilibria of the CSM and the symmetric equilibria of the college admissions problem with no signaling. Under this definition, students' gains are strictly increasing with respect to the student skills, but eventually, all students may get losses depending on the prior distribution of skills. Since colleges' gains require the analysis of order statistics, we consider the particular case of exponential distributed skills which allows us to find closed form formulas of colleges' gains. The exponential model has very interesting implications. First of all, colleges' gains are monotone increasing in college qualities. Second, colleges' gains are monotone increasing in the number of students, i.e. all colleges benefit from an increasing demand for school places. Finally, we show that a sufficiently large demand for school places leads all colleges to get positive gains.

8 Appendices

8.1 Appendix A: The signaling equilibrium

The maximization problem of any student with parameter α is:

$$\max_{P_1 \geq 0} \left\{ \sum_{k=1}^{M} v_k \binom{N-1}{k-1} F\left(\rho_N^{-1}(P_1)\right)^{N-k} \left[1 - F\left(\rho_N^{-1}(P_1)\right)\right]^{k-1} - \frac{c(P_1)}{\phi(\alpha)} \right\} \tag{25}$$

Let's define the function $\varphi(x, N, k) = F(x)^{N-k} [1 - F(x)]^{k-1}$. Hence, for each $k \in \{2, ..., N-1\}$ it is satisfied the following,

$$\varphi'(x, N, k) = [(N-k)(1 - F(x)) - (k-1) F(x)] F(x)^{N-1-k} [1 - F(x)]^{k-2} f(x). \tag{26}$$

Hence, the FOC of the payoff function $\pi(\alpha, P_1)$ with respect to P_1 is given by,

$$\left\{ \begin{array}{c} v_1 (N-1) F\left(\rho_N^{-1}(P_1)\right)^{N-2} \frac{f\left(\rho_N^{-1}(P_1)\right)}{\rho_N'\left(\rho_N^{-1}(P_1)\right)} + \\ \sum_{k=2}^{M} v_k (N-k) \binom{N-1}{k-1} F\left(\rho_N^{-1}(P_1)\right)^{N-1-k} \left[1 - F\left(\rho_N^{-1}(P_1)\right)\right]^{k-1} \frac{f\left(\rho_N^{-1}(P_1)\right)}{\rho_N'\left(\rho_N^{-1}(P_1)\right)} - \\ -\sum_{k=2}^{M} v_k (k-1) \binom{N-1}{k-1} F\left(\rho_N^{-1}(P_1)\right)^{N-k} \left[1 - F\left(\rho_N^{-1}(P_1)\right)\right]^{k-2} \frac{f\left(\rho_N^{-1}(P_1)\right)}{\rho_N'\left(\rho_N^{-1}(P_1)\right)} \end{array} \right\} - \frac{c'(P_1)}{\phi(\alpha)} = 0 \quad (27)$$

In a symmetric equilibrium it is satisfied $P_1 = \rho_M(\alpha)$, then

$$\left\{ \begin{array}{c} v_1 (N-1) \phi(\alpha) F(\alpha)^{N-2} f(\alpha) + \\ + \sum_{k=2}^{M} v_k (N-k) \binom{N-1}{k-1} \phi(\alpha) F(\alpha)^{N-1-k} \left[1 - F(\alpha)\right]^{k-1} f(\alpha) - \\ - \sum_{k=2}^{M} v_k (k-1) \binom{N-1}{k-1} \phi(\alpha) F(\alpha)^{N-k} \left[1 - F(\alpha)\right]^{k-2} f(\alpha) \end{array} \right\} = c'\left(\rho_N(\alpha)\right) \rho_N'(\alpha) \quad (28)$$

By reordering and solving this differential equation with the initial condition $\rho_M(0) = 0$, we find that the signaling strategy $\rho_M(\alpha)$ satisfies,

$$\rho_M(\alpha) = c^{-1} \left(\begin{array}{c} (N-1) \sum_{k=1}^{M-1} (v_k - v_{k+1}) \binom{N-2}{k-1} \int_0^\alpha \phi(x) F(x)^{N-1-k} \left[1 - F(x)\right]^{k-1} f(x)\, dx + ... \\ ... + (N-1) v_M \binom{N-2}{M-1} \int_0^\alpha \phi(x) F(x)^{N-M-1} \left[1 - F(x)\right]^{M-1} f(x)\, dx \end{array} \right) \quad (29)$$

This completes the maximization problem of any student with parameter α. Note that $(N-1)\binom{N-2}{k-1} = \frac{(N-1)!}{(k-1)!(N-1-k)!}$, then we can re-write this signaling strategy as follows,

$$\rho_M(\alpha) = c^{-1} \left(\sum_{k=1}^{M-1} (v_k - v_{k+1}) \int_0^\alpha \phi(x) f_{(k,N-1)}(x)\, dx + v_M \int_0^\alpha \phi(x) f_{(M,N-1)}(x)\, dx \right) \quad (30)$$

Where $f_{(k,N-1)}(x) = \frac{(N-1)!}{(k-1)!(N-1-k)!} F(x)^{N-1-k} \left[1 - F(x)\right]^{k-1} f(x)$ for $k = 1, ..., M$. Note that $f_{(k,N-1)}(x)$ is the density probability function of the x-th highest order statistic from an iid sample of size $N-1$.

Proof of proposition 2:

Consider that any student with parameter α is planing to deviate from the signaling strategy $\rho_M(\alpha)$ by choosing an alternative score P'. Assume w.l.g. that $0 \leq P' < \rho_M(\alpha)$, since the signaling strategy is strictly increasing in α there exists a unique $0 \leq \alpha' < \alpha$ such that $P' = \rho_M(\alpha')$. Then by choosing the score P' the student gets the expected payoff $\pi(\alpha, P') = \pi(\alpha, \rho_M(\alpha'))$ given by,

$$\pi\left(\alpha, \rho_M(\alpha')\right) = \sum_{k=1}^{M} v_k \binom{N-1}{k-1} F(\alpha')^{N-k} \left[1 - F(\alpha')\right]^{k-1} - \frac{c\left(\rho_M(\alpha')\right)}{\phi(\alpha)} \quad (31)$$

By deviating to P', the student losses the extra payoff,

$$\pi\left(\alpha, \rho_M(\alpha)\right) - \pi\left(\alpha, \rho_M(\alpha')\right) = \left\{ \begin{array}{c} \sum_{k=1}^{M} v_k \binom{N-1}{k-1} \left(F(\alpha)^{N-k} \left[1 - F(\alpha)\right]^{k-1} - F(\alpha')^{N-k} \left[1 - F(\alpha')\right]^{k-1} \right) - \\ - \frac{1}{\phi(\alpha)} \left(c\left(\rho_M(\alpha)\right) - c\left(\rho_M(\alpha')\right) \right) \end{array} \right. \quad (32)$$

Note that the increment in the signaling cost $c\left(\rho_M(\alpha)\right) - c\left(\rho_M(\alpha')\right)$ is positive and can be written as,

$$c\left(\rho_M\left(\alpha\right)\right)-c\left(\rho_M\left(\alpha'\right)\right)=\sum_{k=1}^{M-1}\left(v_k-v_{k+1}\right)\int_{\alpha'}^{\alpha}\phi\left(x\right)f_{(k,N-1)}\left(x\right)dx+v_M\int_{\alpha'}^{\alpha}\phi\left(x\right)f_{(M,N-1)}\left(x\right)dx \tag{33}$$

Since $\phi\left(x\right)$ is strictly increasing and positive in x, it is clear that

$$\frac{1}{\phi\left(\alpha\right)}\left(c\left(\rho_M\left(\alpha\right)\right)-c\left(\rho_M\left(\alpha'\right)\right)\right)<\sum_{k=1}^{M-1}\left(v_k-v_{k+1}\right)\int_{\alpha'}^{\alpha}f_{(k,N-1)}\left(x\right)dx+v_M\int_{\alpha'}^{\alpha}f_{(M,N-1)}\left(x\right)dx \tag{34}$$

Note that by reordering the previous equation, we find that the following condition holds,

$$\frac{1}{\phi\left(\alpha\right)}\left(c\left(\rho_M\left(\alpha\right)\right)-c\left(\rho_M\left(\alpha'\right)\right)\right)<\left(N-1\right)v_1\int_0^{\alpha}F\left(x\right)^{N-2}f\left(x\right)dx+\sum_{k=2}^{M}v_k\binom{N-1}{k-1}\int_0^{\alpha}\varphi'\left(x,N,k\right)dx \tag{35}$$

Note that for any $k\in\{2,...,N-1\}$, it holds

$$\int_{\alpha'}^{\alpha}\varphi'\left(x,N,k\right)dx=F\left(\alpha\right)^{N-k}\left[1-F\left(\alpha\right)\right]^{k-1}-F\left(\alpha'\right)^{N-k}\left[1-F\left(\alpha'\right)\right]^{k-1} \tag{36}$$

Which implies that $\pi\left(\alpha,\rho_M\left(\alpha\right)\right)-\pi\left(\alpha,\rho_M\left(\alpha'\right)\right)>0$. By a similar argument, it is possible to show that any deviation $P'>\rho_N\left(\cdot\right)$ cannot be a profitable deviation. This completes the proof.

Proof of Proposition 3:

- $\rho_M\left(\alpha\right)$ is strictly increasing in α and bounded above.

To prove that the signaling strategy $\rho_M\left(\alpha\right)$ is strictly increasing in α, it is enough to show that the function $c\left(\rho_M\left(\alpha\right)\right)$ is strictly increasing in α, since $c\left(\cdot\right)$ is a strictly increasing function. Then

$$\frac{d}{d\alpha}\left(c\left(\rho_M\left(\alpha\right)\right)\right)=\left\{\begin{array}{c}\left(N-1\right)\sum_{k=1}^{M-1}\left(v_k-v_{k+1}\right)\binom{N-2}{k-1}\phi\left(\alpha\right)F\left(\alpha\right)^{N-1-k}\left[1-F\left(\alpha\right)\right]^{k-1}f\left(\alpha\right)+...\\ ...+\left(N-1\right)v_M\binom{N-2}{M-1}\phi\left(\alpha\right)F\left(\alpha\right)^{N-M-1}\left[1-F\left(\alpha\right)\right]^{M-1}f\left(\alpha\right)\end{array}\right. \tag{37}$$

It is clear that $\frac{d}{d\alpha}\left(c\left(\rho_M\left(\alpha\right)\right)\right)>0$ for all α, as we desired. To prove that signaling strategy $\rho_M\left(\alpha\right)$ is bounded above, we use the the fact that this function can be written as follows,

$$c\left(\rho_M\left(\alpha\right)\right)=\sum_{k=1}^{M-1}\left(v_k-v_{k+1}\right)\int_0^{\alpha}\phi\left(x\right)f_{(k,N-1)}\left(x\right)dx+v_M\int_0^{\alpha}\phi\left(x\right)f_{(M,N-1)}\left(x\right)dx \tag{38}$$

Where $f_{(k,N-1)}\left(x\right)$ is the density function of the $k-th$ order statistic from an $N-1$ sample with distribution function $F\left(x\right)$ such that $x_{(1,N-1)}=\underset{1\leq i\leq N-1}{max}\{x_i\}$, $x_{(2,N-1)}$=second greatest in $\{x_i\}_{i=1}^{N-1}$ and so on. Since $\phi\left(x\right)$ is strictly increasing and bounded in $[0,w]$, we know that

$$c\left(\rho_M\left(\alpha\right)\right)\leq\phi\left(w\right)\left(\sum_{k=1}^{M-1}\left(v_k-v_{k+1}\right)\int_0^w f_{(k,N-1)}\left(x\right)dx+v_M\int_0^w f_{(M,N-1)}\left(x\right)dx\right) \quad (39)$$

But by definition $\int_0^w f_{(k,N-1)}\left(x\right)dx=1$ for all $k=1,...,M$. Then

$$c\left(\rho_M\left(\alpha\right)\right)\leq\phi\left(w\right)\left(\sum_{k=1}^{M-1}\left(v_k-v_{k+1}\right)+v_M\right)<\infty \quad (40)$$

- $\pi\left(\alpha,\rho_M\left(\alpha\right)\right)$ is strictly increasing in α .

Now we want to show that the equilibrium payoff $\pi\left(\alpha,\rho_M\left(\alpha\right)\right)$ is strictly increasing in α. To prove this property, we calculate the derivative of the payoff function with respect to α. Then

$$\frac{d}{d\alpha}\left(\pi\left(\alpha,\rho_M\left(\alpha\right)\right)\right)=\begin{cases}\left(N-1\right)v_1F\left(\alpha\right)^{N-2}f\left(\alpha\right)+\sum\limits_{k=2}^{M}v_k\binom{N-1}{k-1}\varphi'\left(\alpha\right)-\\-\frac{1}{\phi(\alpha)^2}\left(\phi\left(\alpha\right)\frac{d}{d\alpha}\left(c\left(\rho_M\left(\alpha\right)\right)\right)-c\left(\rho_M\left(\alpha\right)\right)\phi'\left(\alpha\right)\right)\end{cases} \quad (41)$$

By reordering the previous expression, it is easy to show that

$$\frac{d}{d\alpha}\left(\pi\left(\alpha,\rho_M\left(\alpha\right)\right)\right)=\frac{c\left(\rho_M\left(\alpha\right)\right)\phi'\left(\alpha\right)}{\phi\left(\alpha\right)^2}>0 \quad (42)$$

This completes the proof.

8.2 Appendix B: Comparative statics

Proof of Proposition 4:

Let $\rho_M\left(\alpha,N\right)$ be the equilibrium signaling strategy of any college admissions problem with $M\geq1$ colleges and $N>M$ students. Since the cost function $c\left(\cdot\right)$ is strictly increasing, it is enough to show that the function $c\left(\rho_M\left(\alpha,N\right)\right)$ satisfies the desired properties. Then, it is easy to show that the difference $c\left(\rho_M\left(\alpha,N+1\right)\right)-c\left(\rho_M\left(\alpha,N\right)\right)$ is equal to

$$\begin{cases}\sum\limits_{k=1}^{M-1}\left(v_k-v_{k+1}\right)\int_0^\alpha\left(N\binom{N-1}{k-1}F\left(x\right)-\left(N-1\right)\binom{N-2}{k-1}\right)\phi\left(x\right)F\left(x\right)^{N-1-k}\left[1-F\left(x\right)\right]^{k-1}f\left(x\right)dx+...\\...+v_M\int_0^\alpha\left(N\binom{N-1}{M-1}F\left(x\right)-\left(N-1\right)\binom{N-2}{M-1}\right)\phi\left(x\right)F\left(x\right)^{N-M-1}\left[1-F\left(x\right)\right]^{M-1}f\left(x\right)dx\end{cases} \quad (43)$$

Given that $\left(N-1\right)\binom{N-2}{k-1}=\left(N-k\right)\binom{N-1}{k-1}$, the previous equation reduces to the following,

$$\begin{cases}\sum\limits_{k=1}^{M-1}\left(v_k-v_{k+1}\right)\binom{N-1}{k-1}\int_0^\alpha\left(NF\left(x\right)-\left(N-k\right)\right)\phi\left(x\right)F\left(x\right)^{N-1-k}\left[1-F\left(x\right)\right]^{k-1}f\left(x\right)dx+...\\...+v_M\binom{N-1}{M-1}\int_0^\alpha\left(NF\left(x\right)-\left(N-M\right)\right)\phi\left(x\right)F\left(x\right)^{N-M-1}\left[1-F\left(x\right)\right]^{M-1}f\left(x\right)dx\end{cases} \quad (44)$$

Then, it is clear that $c\left(\rho_M\left(\alpha,N+1\right)\right)-c\left(\rho_M\left(\alpha,N\right)\right)<0$ if $\alpha\leq\alpha_N\left(N\right)$ where $\alpha_N\left(N\right)$ is the unique solution to the equation,

$$F(x) = 1 - \frac{M}{N} \tag{45}$$

Further, the threshold $\alpha_N(N)$ is monotone increasing in N, i.e. $\alpha_N(N+1) > \alpha_N(N)$ for all $N > M$. This completes the proof.

Proof of Proposition 5:

Consider the equilibrium signaling strategy of any college admissions problem with one college with $M \geq 2$ school seats and $N \geq M+2$ students.

$$\rho_M(\alpha, M) = c^{-1}\left((N-1)\, v_1 \tbinom{N-2}{M-1} \textstyle\int_0^\alpha \phi(x)\, F(x)^{N-1-M} [1-F(x)]^{M-1} f(x)\, dx \right) \tag{46}$$

Since the function $c(\cdot)$ is strictly increasing, to prove this result we focus on the function $c(\rho_M(\alpha, M))$. Then, it is easy to show that the difference $c(\rho_M(\alpha, M+1)) - c(\rho_M(\alpha, M))$ is equal to

$$v_1(N-1)\int_0^\alpha \left(\binom{N-2}{M}(1-F(x)) - \binom{N-2}{M-1}F(x)\right)\phi(x)\, F(x)^{N-2-M}[1-F(x)]^{M-1} f(x)\, dx \tag{47}$$

By reordering and applying the identity $\binom{N}{k} = \binom{N-1}{k-1} + \binom{N-1}{k}$, the previous equation reduces to the following,

$$v_1(N-1)\int_0^\alpha \left(\binom{N-2}{M} - \binom{N-1}{M}F(x)\right)\phi(x)\, F(x)^{N-2-M}[1-F(x)]^{M-1} f(x)\, dx \tag{48}$$

Given that $\binom{N-1}{M} = \frac{N-1}{N-1-M}\binom{N-2}{M}$, we get

$$v_1(N-1)\binom{N-2}{M}\int_0^\alpha \left(1 - \frac{N-1}{N-1-M}F(x)\right)\phi(x)\, F(x)^{N-2-M}[1-F(x)]^{M-1} f(x)\, dx \tag{49}$$

Then, it is clear that $c(\rho_M(\alpha, M+1)) - c(\rho_M(\alpha, M)) > 0$ if $\alpha \leq \alpha_N(M, N)$ where $\alpha_M(M, N)$ is the unique solution to the equation,

$$F(x) = 1 - \frac{N-1-M}{N-1} \tag{50}$$

Further, it is clear that the threshold $\alpha_M(M, N)$ is monotone increasing in N and monotone decreasing in M. This completes the proof.

Proof of Proposition 6:

We analyze the effect of a change in the quality of the college c_k, then we consider that the equilibrium signaling strategy depends on the quality of this college, i.e. $\rho_M(\alpha, v_k)$. We know that the equilibrium signaling strategy satisfies the equation,

$$c(\rho_M(\alpha, v_k)) = (N-1)\, v_1 \int_0^\alpha \phi(x)\, F(x)^{N-2} f(x)\, dx + \sum_{k=2}^{M} v_k \binom{N-1}{k-1}\int_0^\alpha \phi(x)\, \varphi'(x, N, k)\, dx \tag{51}$$

Consider a change in the quality of the college c_k such that $v_{k-1} > v'_k > v_{k+1}$, i.e. a change in college qualities that preserve students' ordinal preferences. It is not difficult to show that the difference $c(\rho_M(\alpha, v'_k)) - c(\rho_M(\alpha, v_k))$ satisfies the following equation for $k = 2, ..., M$,

$$c\left(\rho_M\left(\alpha, v'_k\right)\right) - c\left(\rho_M\left(\alpha, v_k\right)\right) = \left(v'_k - v_k\right) \binom{N-1}{k-1} \int_0^{\alpha} \phi(x) \varphi'(x, N, k)\, dx \qquad (52)$$

Since $\varphi'(x, N, k) = [(N-k)(1-F(x)) - (k-1)F(x)] F(x)^{N-1-k} [1-F(x)]^{k-2} f(x)$, it is easy to show that whenever $\alpha \le \alpha_{v_k}(N, k)$

$$sgn\left(c\left(\rho_M\left(\alpha, v'_k\right)\right) - c\left(\rho_M\left(\alpha, v_k\right)\right)\right) = sgn\left(v'_k - v_k\right) \qquad (53)$$

Where $\alpha_{v_k}(N, k)$ is the unique solution to the equation,

$$F(x) = \frac{N-k}{N-1} \qquad (54)$$

Further, it is easy to observe that the threshold $\alpha_{v_k}(N, k)$ is monotone increasing in N for all $k = 2, ..., M$ and monotone decreasing in k. This completes the proof.

8.3 Appendix C: Gains of the CSM

If α is distributed according to an exponential distribution function $f(\alpha; \theta) = \frac{1}{\theta} e^{-\frac{\alpha}{\theta}}$ for $\alpha \in [0, \infty)$ and $\theta > 0$, then

1. $E[\alpha] = \theta$ and
2. $E\left[\alpha_{(j)}\right] = \sum_{k=1}^{N+1-j} \frac{\theta}{N+1-k}$ for $j = 1, ..., N$.

where $\alpha_{(1)} = \max_{1 \le i \le N} \alpha_i$, $\alpha_{(2)}$=second greatest in $\{\alpha_i\}_{i=1}^{N-1}$and so on (Huang, 1974). Consider any M by N college admissions problem such that $N > M \ge 1$, then colleges' gains satisfy,

$$\Delta EQ(j, N) = \theta \sum_{k=1}^{N+1-j} \frac{1}{N+1-k} - \theta \left(\frac{N-1}{N}\right)^{j-1}. \qquad (55)$$

Assume w.l.g. that $\theta = 1$. We establish the following auxiliary results.

Claim 1 *The continuous function $f(x) = \left(\frac{x-1}{x}\right)^{j-1}$ is strictly increasing and strictly concave in x for all $x > j \ge 3$.*

Proof. To prove this result, we simply take the first and second derivative of the function $f(x) = \left(\frac{x-1}{x}\right)^{j-1}$. Then it is easy to show the following:

1. $f'(x) = \left(\frac{j-1}{x^2}\right)\left(\frac{x-1}{x}\right)^{j-2} > 0$; and
2. $f''(x) = \left(\frac{j-1}{x^4}\right)\left(\frac{x-1}{x}\right)^{j-3}(j - 2x) < 0$.

For all $x > j \ge 3$, this completes the proof. ■

Lemma 1 $\Delta EQ(j,N) > \Delta EQ(j+1,N)$ *for all* $j = 1,...,M-1$.

Proof. It is not difficult to show that the difference $\Delta EQ(j,N) - \Delta EQ(j+1,N)$ satisfies the following,

$$\Delta EQ(j,N) - \Delta EQ(j+1,N) = \frac{1}{j} - \left(\frac{N-1}{N}\right)^{j-1}\left(\frac{1}{N}\right) \tag{56}$$

Since, $\left(\frac{N-1}{N}\right)^{j-1} \leq 1$ for all $j \geq 1$, we know that

$$\Delta EQ(j,N) - \Delta EQ(j+1,N) \geq \frac{1}{j} - \frac{1}{N} = \frac{N-j}{jN}. \tag{57}$$

Since $N > M \geq j$, $\Delta EQ(j,N) - \Delta EQ(j+1,N) > 0$ for $j = 1,...,M-1$. This completes the proof. ■

Lemma 2 $\Delta EQ(j,N)$ *is strictly monotone increasing in* $N > M$ *for all* $j = 1,...,M$.

Proof. Consider the following function for a given $j = 1,...,M$,

$$\Delta EQ(j,N+1) - \Delta EQ(j,N) = \begin{cases} \sum_{k=1}^{N+2-j} \frac{1}{N+2-k} - \sum_{k=1}^{N+1-j} \frac{1}{N+1-k} - \\ -\left[\left(\frac{N}{N+1}\right)^{j-1} - \left(\frac{N-1}{N}\right)^{j-1}\right] \end{cases} \tag{58}$$

By simplifying, we can get the following expression,

$$\Delta EQ(j,N+1) - \Delta EQ(j,N) = \frac{1}{N+1} - \left(\left(\frac{N}{N+1}\right)^{j-1} - \left(\frac{N-1}{N}\right)^{j-1}\right) \tag{59}$$

It is not difficult to show by a direct inspection that $\Delta EQ(j,N+1) - \Delta EQ(j,N) > 0$ for $j = 1,2$. Now consider the case of any j such that $N > M \geq j \geq 3$. By the Claim 1, we know that

$$f'(N) \geq \left(\frac{N}{N+1}\right)^{j-1} - \left(\frac{N-1}{N}\right)^{j-1} \tag{60}$$

where $f(x) = \left(\frac{x-1}{x}\right)^{j-1}$ such that $x > j \geq 3$. Hence,

$$\Delta EQ(j,N+1) - \Delta EQ(j,N) \geq \frac{1}{N+1} - \left(\frac{j-1}{N^2}\right)\left(\frac{N-1}{N}\right)^{j-2} \tag{61}$$

Since $\left(\frac{N-1}{N}\right)^{j-2} \leq 1$ for all $j \geq 2$, we know that

$$\Delta EQ(j,N+1) - \Delta EQ(j,N) \geq \frac{1}{N+1} - \frac{j-1}{N^2} = \frac{N^2 - (j-1)(N+1)}{N^2(N+1)}. \tag{62}$$

Given that $N > M \geq j \geq 3$ and $(N-1)(N+1) > (j-1)(N+1)$, we conclude that

$$\Delta EQ(j,N+1) - \Delta EQ(j,N) > \frac{1}{N^2(N+1)}. \tag{63}$$

Then $\Delta EQ(j,N+1) - \Delta EQ(j,N) > 0$ for all $N > M \geq j \geq 3$. This completes the proof. ■

Proof of Proposition 7

Properties 1 and 2 of colleges gains $\Delta EQ\,(j,N)$ come directly from Lemmas 1 and 2. For the third property, assume that $\Delta EQ\,(M,N) \geq 0$ for $N = M+1$, then $N^* = M+1$. By Lemma 2, $\Delta EQ\,(M,N) \geq 0$ for all $N \geq N^* > M$. By Lemma 1, $\Delta EQ\,(j,N) \geq 0$ for all $j = 1, ..., M$ provided $\Delta EQ\,(M,N) \geq 0$. Then $\Delta EQ\,(j,N) \geq 0$ for all $N \geq N^*$ and $j = 1, ..., M$.

Now suppose that $\Delta EQ\,(M,N) < 0$ for $N = M+1$. Note that,

1. $\lim\limits_{N\to\infty} \left(\frac{N-1}{N}\right)^{M-1} = 1$ for all $M \geq 1$; and

2. $\lim\limits_{N\to\infty} E\left[\alpha_{(M)}\right] = \lim\limits_{N\to\infty} \sum\limits_{k=1}^{N+1-M} \frac{1}{N+1-k} = \infty.$

Then there exists a $N^* > M$ such that $\Delta EQ\,(M,N^*) \geq 0$. Then by Lemmas 1 and 2, $\Delta EQ\,(j,N) \geq 0$ for all $N \geq N^*$ and $j = 1, ..., M$. This completes the proof.

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